Geometry, Proofs, and the Common Core Standards

Sue Olson, Ed.D.
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Structure

• Presentation of representative proof problem
• Solution
• Notes on task/modifications
Geometry – Congruence
G.CO.10

• Prove Theorems about triangles
  • The sum of the measures of the interior angles of a triangle is 180°.
  • The base angles of an isosceles triangle are congruent.
  • The segment joining the midpoints of two sides of a triangle is parallel to the third side and half the length of the third side.
  • The medians of a triangle meet at a point.
Given: Triangle ABC
Prove: $m\angle A + m\angle B + m\angle C = 180^\circ$
## Solution

Given: Triangle ABC  
Prove: $m \angle CAB + m \angle ABC + m \angle ACB = 180^\circ$

<table>
<thead>
<tr>
<th>Statement</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Triangle ABC</td>
<td>1. Given</td>
</tr>
<tr>
<td>2. Draw line XY through A and parallel to segment BC</td>
<td>2. Through a point not on a line there is exactly one line parallel to the given line. (Parallel Postulate)</td>
</tr>
<tr>
<td>3. $\angle XAC \cong \angle C$, $\angle YAB \cong \angle B$</td>
<td>3. If two parallel lines are cut by a transversal, then the alternate interior angles are congruent.</td>
</tr>
<tr>
<td>4. $m \angle XAC \cong m \angle C$, $m \angle YAB \cong m \angle B$</td>
<td>4. If angles are congruent, then they have equal measures.</td>
</tr>
<tr>
<td>5. $m \angle CAB + m \angle YAB + m \angle XAC = 180^\circ$</td>
<td>5. If three or more adjacent angles form a straight angle, their sum is $180^\circ$</td>
</tr>
<tr>
<td>$\therefore 6. m \angle CAB + m \angle ABC + m \angle ACB = 180^\circ$</td>
<td>6. Substitution property of equality.</td>
</tr>
</tbody>
</table>
Notes/Modifications

• This proof requires that students know parallel line theorems and the Parallel Postulate.
Notes/Modifications

• The deductive proof can be made easier in the following ways:
  • Provide a hint to draw a parallel line through vertex A, parallel to side BC.
  • Provide solution drawing.
  • Provide solution drawing and all statements.
  • Provide solution drawing and all statements and a mixed up list of reasons (extraneous reasons could also be supplied...other parallel theorems, etc.)
An inductive experience

1. Have each student draw a random triangle (suggest each row do a different type so that you are reviewing vocabulary…scalene, right, etc.). Using a protractor students measure each angle and then find the sum of the angles. Classroom develops conjecture about the sum of the angles of a triangle.
An inductive experience

2. Provide random cut out triangles to each student in the class. Have students measure sides and angles so that they can name the triangle and write the sum of the angles. Classroom develops conjecture about the sum of the angles of a triangle.
An inductive experience

3. Again, using cut out triangles, have students locate midpoint (M) of one side (BC). Fold vertex B and C so that they meet at M, crease folds. Fold A so that it meets at M also. The three angles A, B, and C should meet at M forming a straight angle. They should be adjacent angles with only their edges touching. Discuss how this demonstrates that the sum of the angles of a triangle is 180°.
Geometry – Congruence
G.CO.11

• Prove Theorems about parallelograms
  • The opposites sides of a parallelogram are congruent.
  • The diagonals of a parallelogram bisect each other.
  • The opposite angles of a parallelogram are congruent.
  • If a parallelogram has congruent diagonals, then it is a rectangle.
Parallelograms

Given: Parallelogram ABCD

Prove: \( AB \cong CD \) and \( AD \cong BC \)

Select the correct statements from the list to complete the proof.

<table>
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<tbody>
<tr>
<td>1. Parallelogram ABCD</td>
<td></td>
</tr>
<tr>
<td>2. ( AB \parallel CD, AD \parallel BC )</td>
<td></td>
</tr>
<tr>
<td>3. ( \angle 1 \cong \angle 4, \angle 2 \cong \angle 3 )</td>
<td></td>
</tr>
<tr>
<td>4. ( \overline{BD} \cong \overline{DB} )</td>
<td></td>
</tr>
<tr>
<td>5. ( \triangle ABD \cong \triangle CDB )</td>
<td></td>
</tr>
<tr>
<td>( \therefore 6. \overline{AB} \cong \overline{DC}, \overline{AD} \cong \overline{BC} )</td>
<td></td>
</tr>
</tbody>
</table>

Thus we have proved the following theorem.

**If a quadrilateral is a parallelogram then its opposite sides are congruent.**
The List....select the appropriate answers for the proof from the list and write the correct answer in the space provided.

1. If the opposite sides of a quadrilateral are parallel, then it is a parallelogram.
2. If a quadrilateral is a parallelogram, then its opposite sides are parallel.
3. If two sides and an included angle of one triangle are congruent to two corresponding sides and an included angle of another triangle, then the triangles are congruent. (SAS)
4. If three sides of one triangle are congruent to three sides of another triangle, then the triangles are congruent. (SSS)
5. If two angles and an included side of one triangle are congruent to two angles and an included side of another triangle, then the triangles are congruent. (ASA)
6. If the hypotenuse and leg of one right triangle are congruent to the hypotenuse and leg of another right triangle, then the triangles are congruent. (HL)
7. Given
8. Every segment is congruent to itself. (Reflexive Property of Congruence)
9. If two triangles are congruent, then all of their corresponding sides and angles are congruent. (CPCTC)
10. If lines are parallel, then alternate interior angles are congruent.
11. If alternate interior angles are congruent, then the lines are parallel.
12. If lines are parallel, then the alternate exterior angles are congruent.
13. If alternate exterior angles are congruent, then the lines are parallel.
14. If lines are parallel, then the corresponding angles are congruent.
15. If corresponding angles are congruent, then the lines are parallel.
10. If lines are parallel, then alternate interior angles are congruent.
11. If alternate interior angles are congruent, then the lines are parallel.
12. If lines are parallel, then the alternate exterior angles are congruent.
13. If alternate exterior angles are congruent, then the lines are parallel.
14. If lines are parallel, then the corresponding angles are congruent.
15. If corresponding angles are congruent, then the lines are parallel.
Solution:
1. Given
2. If a quadrilateral is a parallelogram, then its opposite sides are parallel (definition)
3. If lines are parallel, then the alternate interior angles are congruent.
4. Reflexive Property of congruence.
5. ASA (3,2,3)
6. CPCTC
Notes on this task

• Easier – randomly list just the six steps for the solution.
• Grouping of theorems models “chunking”.
• List requires students to discriminate between the conditional and converse to determine proper logic flow.
Parallelogram Example 2
Parallelograms

Given: Parallelogram ABCD
Prove: \( \angle A \cong \angle C, \angle B \cong \angle D \)

Select the correct statements from the list to complete the proof.

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<td>1. Parallelogram ABCD</td>
<td></td>
</tr>
<tr>
<td>2. ( AB \parallel CD, AD \parallel BC )</td>
<td></td>
</tr>
<tr>
<td>3. Draw BD</td>
<td></td>
</tr>
<tr>
<td>4. ( \angle ABD \cong \angle BDC, \angle ADB \cong \angle DBC )</td>
<td></td>
</tr>
<tr>
<td>5. ( BD \cong DB )</td>
<td></td>
</tr>
<tr>
<td>6. ( \triangle ABD \cong \triangle CDB )</td>
<td></td>
</tr>
<tr>
<td>( \therefore 7. \angle A \cong \angle C )</td>
<td></td>
</tr>
<tr>
<td>( \therefore 8. \angle ABC \cong \angle ADC )</td>
<td></td>
</tr>
</tbody>
</table>

Thus we have proved the following theorem.

If a quadrilateral is a ______________ then its opposite _____ are ____________.
Notes/Modifications

• Easier – select only correct reasons but randomize them
• Keep the list but arrange chunks in the order they would be encountered
• Draw the auxiliary line
Notes/Modifications

- More challenging
  - Remove the statements of the proof but keep the list
  - Remove the statements of the proof as well as the list
  - Make it a diagramless proof.
Turn this into a puzzle proof

• Remove the numbers for the steps of the proof and reasons
• Copy the proof on colored card stock
• Cut off the statements and reasons, cut them into separate pieces and perhaps add a few distractors
• Code all the pieces for one set on the back and place them into an envelope with the code also on the envelope
Turn this into a puzzle proof

- Pair students appropriately, give them an envelope and have them figure out the proof.
- Ask groups that finish early if there is another order for the statements that would still be logical
- Provide them with a paper with the set-up so they can write the correct proof and add to their notes
- Return pieces to the envelope and give them a more challenging one to try.
Parallelograms

Given: Parallelogram ABCD

Prove: AC and BD bisect each other.

Statement    Reason

Thus the following theorem has been proved: ______________________
___________________________________________________________
Notes/Modifications

• This proof is based on the assumption that only the definition of a parallelogram is known and no other parallelogram properties.

• It is easier if you build on the theorem, If a quadrilateral is a parallelogram then its opposite sides are congruent.

• Students are asked to write all statements and reasons in a logical order
Notes/Modifications

- Easier - Supply statements and students supply reasons
- Challenging - make the proof diagramless
- Challenging - make the proof analytic
Parallelogram – Example 4
Parallelograms – Proving related theorems

Missing diagram problem.

*If a parallelogram has congruent diagonals, then it is a rectangle.*

- Draw and label the diagram.
- Write the given and prove statements as they relate to your diagram.
- Provide a deductive proof.
Assume the following facts:

**Definition:** If a quadrilateral is a parallelogram then both pairs of opposite sides are parallel.

**Theorem:** If a quadrilateral is a parallelogram then both pairs of opposite sides are congruent.

**Theorem:** If a quadrilateral is a parallelogram, then both pairs of opposite angles are congruent.

**Theorem:** If a quadrilateral is a parallelogram, then consecutive angles are supplementary.

**Theorem:** If a quadrilateral is a parallelogram, then the diagonals bisect each other.

**Definition:** If a quadrilateral is a rectangle, then it is a parallelogram with at least one right angle.
Notes/Modification

- This is a challenging form of a proof.
- Students must create the set-up as well as write a deductive proof.
- They must understand that in a conditional sentence, the hypothesis is the Given and the conclusion is the Prove.
- They must also understand how to word the definition correctly for logical flow.
Notes/Modification

- Easier-
  - Supply diagram
  - Provide statements or reasons
  - Provide list of possible answers
- More challenging
  - Write the statement in declarative form.
Triangles – Example #2
Given: Isosceles triangle ABC with base BC
Prove: \( \angle B \cong \angle C \)
Given: Isosceles triangle ABC with base BC
Prove: $\angle B \cong \angle C$

<table>
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</tr>
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<tbody>
<tr>
<td>1. Isosceles triangle ABC with base BC</td>
<td>1. Given</td>
</tr>
<tr>
<td>2. $AB \cong AC$</td>
<td>2. If a triangle is isosceles, then it has at least two congruent sides.</td>
</tr>
<tr>
<td>3. Construct median AM</td>
<td>3. Through a vertex of a triangle there is exactly one segment that can be drawn to the midpoint of the opposite side.</td>
</tr>
<tr>
<td>4. $BM \cong MC$</td>
<td>4. If a segment is a median, then it goes from the vertex of a triangle and divides the opposite side into two congruent segments.</td>
</tr>
<tr>
<td>5. $AM \cong AM$</td>
<td>5. Every segment is congruent to itself. (Reflexive property of congruence)</td>
</tr>
<tr>
<td>6. $\triangle ABM \cong \triangle ACM$</td>
<td>6. SSS congruence (1,4,5)</td>
</tr>
<tr>
<td>7. $\angle B \cong \angle C$</td>
<td>7. If two triangles are congruent, then all corresponding sides and angles are congruent. (CPCTC)</td>
</tr>
</tbody>
</table>
Notes/Modifications

• Students must know the definition of an isosceles triangle and a median

• Easier-
  • Provide a hint to draw median from A
  • Provide solution drawing
  • Provide solution drawing and all statements
  • Provide solution drawing, all statements, and a list of possible reasons

• Challenging
  • Prove isosceles triangle ABC congruent to its reflection, triangle ACB
Triangles – Example #3
Given: H is the midpoint of \( \overline{GJ} \)
M is the midpoint of \( \overline{GK} \)

Prove: \( \overline{HM} \parallel \overline{JK} \)
\( HM = \frac{1}{2} JK \)
Notes/Modifications

• This proof is probably best done with all statements written and then either completed together as a class or individually.
• Easier
  • Provide selected reasons to choose from
• Challenging
  • Do as an analytic proof
Notes/Modifications

Use the methods of coordinate geometry to prove that the segment connecting the midpoints of any triangle is parallel to the third side and has a length that is one-half the length of the third side. (diagramless proof) (most challenging)
Use the methods of coordinate geometry to prove that the segment connecting the midpoints of a triangle with vertices A \((2b, 2c)\), B\((2a, 0)\), C\((0,0)\) is parallel to the third side and has a length that is one-half the length of the third side. (diagramless proof or provide diagram) (a little easier)
• Use the methods of coordinate geometry to prove that the segment connecting the midpoints of a triangle with vertices A (8, 10), B(14, 0), C(0,0) is parallel to the third side and has a length that is one-half the length of the third side. Start by drawing a diagram. (easiest but not a true proof, just a demonstration)
Triangles – Example #4
Given: Triangle ABC
Prove: The medians of triangle ABC are concurrent at a point that is two thirds of the way from any vertex of triangle ABC to the midpoint of the opposite side.
Solution

• The equations of the medians are

\[ \overline{AM} : \left(2c \right) \frac{x}{2b-a} - \frac{6ac}{2b-a} \]
\[ \overline{BN} : y = \left(\frac{c}{a+b}\right)x \]
\[ \overline{CP} : \left(\frac{c}{b-2a}\right)x - \frac{6ac}{b-2a} \]

By solving, the intersection is: \((2a+2b, 2c)\)

\[ BI = \frac{2}{3} BN \quad CI = \frac{2}{3} CP \quad AI = \frac{2}{3} AM \]
Notes

• The coordinate proof, taken from *Geometry for Enjoyment and Challenge*, Rhoad, Milauska, Whipple, McDougal, Littell & Co., Evanston, Illinois, 1991 p. 665 is a challenging analytic proof. Students need to understand how to work with algebra equations with many variables, solve systems of equations, calculate midpoint coordinates and determine the length of segments algebraically. They need to know the meaning of concurrent, the definition of a median, etc.
Notes/Modifications

- If an analytic type of proof is desired it might be easier to try triangles with specific numeric vertices. Use different coordinates for each student and have class develop the conjectures based on their observations.
• Inductive: Using dynamic geometry software have students do the following:
  • Create a random triangle.
  • Create midpoints of each side
  • Draw 2 segments from vertex to midpoint. (median)
  • Label coordinates of point of intersection.
  • Repeat for each pair of medians to demonstrate that the intersections are the same and thus concurrent.
  • Measure length of shortest and longest segment of each median.
  • Determine ratio of long segment to short segment of each median.
  • Write a conjecture based on the observations of data from several different triangles.
Notes/Modifications

• Inductive: Pass out papers with large triangles of different shapes to class
  • Have students find midpoints of each side by measuring or folding.
  • Have students draw or fold all medians. Discuss what they observe (medians are concurrent).
  • Have students measure long and short segments of each median and then determine the ratio. Discuss what they observe (medians point of concurrency is 2/3 of the way from any vertex to the midpoint of the opposite side).
  • With class or in small groups have students write a conjecture based on the observations of data from several different triangles.
Mathematical Practices
1. Apply geometric concepts in modeling situations
2. Mathematical Practices
3. Make sense of problems and persevere in solving them.
4. Reason abstractly and quantitatively.
5. Construct viable arguments and critique the reasoning of others.
7. Use appropriate tools strategically.
8. Attend to precision.
9. Look for and make use of structure.
10. Look for and express regularity in repeated reasoning.
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Harvard-Westlake Geometry Team Production

\[ x^2 + y^2 + 2dx + 2ey + f = 0 \]

\[ (x,y) = F(x',y') \]

\[ a = \pi r^2 \]