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*Strengthening the Emphasis on Mathematical Modeling*
Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society and the workplace. In early grades, this might be as simple as writing an addition equation to describe a situation. In middle grades, a student might apply proportional reasoning to plan a school event or analyze a problem in the community. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.
How is this interpretation of modeling stronger than what is emphasized in current or past practice?

- Levels of modeling (grade level and level of integration)
- Mathematical tools (CCSM emphasizes building on previous knowledge)
- Practical assumptions (approximate)
- Limitations of model (conceptual)
- Representation of quantitative relationships (theoretical, concrete)
- Interpretation of results (consistent, applicable, predictive)
An problem from combinatorics:

A circuit is made out of wires and nodes (places where the wires are joined). Can you design a circuit where the number of nodes on each wire is equal to the number of wires joined at each node?

Manipulatives? Graphs? Tables or matrices?
"Smallest" case?
Must the number of wires equal the number of nodes?
Is any number of wires and nodes possible?
The classic "work" problem:

A can paint a room in 2 hours. B can paint a room in 3 hours. How long will it take them to paint a room if they work together?

**Basic assumption:** Effort is scalable, e.g., working together does not introduce hindrance, so should shorten job time.

**Strategy:** Interpret A and B as rates and use as variables.

$$\text{rooms/hour} \times \text{hours} = \text{rooms}$$

$$A \times 1 = \frac{1}{2}$$

$$B \times 1 = \frac{1}{3}$$

**Model:** Rates are additive because of basic assumption, i.e., we choose to interpret A + B as the rate for working together.

$$(A + B) \times 1 = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$$

1 h12 min to paint room

**Is model reasonable and consistent?**

If B also paints at rate of $\frac{1}{2}$ room per hour then it takes 1 hour to paint the room

$$\frac{1}{2} + \frac{1}{2} = 1$$

i.e., equal rates of work produce results consistent with our basic assumption.

So if B is much slower than A

$$\frac{1}{2} + \frac{1}{n} - \frac{1}{2}$$

the advantage of including B becomes negligible. (Miller’s Law: Look babe, if A can paint a room in 2 hours and B can paint a room in 16 hours why doesn’t B just go home and let A paint the room?)

**Predictive:** Rates add as reciprocals of time, like resistances in a circuit.

A more "realistic" model: When two painters work together each rate is diminished in proportion to the difference between their two rates $r_1$ and $r_2$.

$$R_1 + R_2 = (r_2 - r_1 + 1)(r_1 + r_2) \leq r_1 + r_2$$

$$r_2 \leq r_1$$

**Note:** By this model it will take A and B

1 h26 min24 s

to paint the room together.
A "simpler" rate problem:

At an exhibition game a baseball player throws his fastest pitch (100mph) to the catcher from a vehicle driving straight toward home plate at 25mph as measured by a radar unit in the bleachers. What speed does the radar measure for the baseball?

What would Galileo say?

\[ 100 \text{mph} + 25 \text{mph} = 125 \text{mph} \]

NASA measures a space probe leaving the solar system at 100 kps. The probe fires off a projectile in the same direction of motion at 25 kps as measured by its on-board computer. What is the speed of the projectile as measured by NASA?

Galilean model: \[ 100 \text{kps} + 25 \text{kps} = 125 \text{kps} \]

What would Einstein say? \( c = 3 \times 10^5 \text{kps} \)

\[
\begin{align*}
    v_1 &= \frac{100}{c} \\
    v_2 &= \frac{25}{c} \\
    v_N &= \frac{v_1 + v_2}{1 + v_1 v_2} \\
    cv_N &= \frac{4500 \, 000 \, 000}{36 \, 000 \, 001} \text{kps}
\end{align*}
\]

By how much do these two models differ?

\[
125 - \frac{4500 \, 000 \, 000}{36 \, 000 \, 001} = \frac{125}{36 \, 000 \, 001} \approx 3.472 \times 10^{-6} \text{kps} \approx .0125 \text{kph}
\]

A sub-atomic particle travels at \( 2.4 \times 10^5 \text{kps} \) as measured by the control room at a linear accelerator. The particle emits a pion at \( 1.5 \times 10^5 \text{kps} \) relative to its frame of reference, in the same direction of travel. What is the speed of the pion as measured by the control room?

Galileo is in big trouble: \( v_G = (2.4 \times 10^5 \text{kps}) + (1.5 \times 10^5 \text{kps}) = 3.9 \times 10^5 \text{kps} \gg c \)
Einstein? $v_1 = \frac{3 \cdot 4}{3} = 0.8, \ v_2 = \frac{1 \cdot 5}{3} = 0.5$

\[
\begin{align*}
v &= \frac{0.8 + 0.5}{1 + (0.8)(0.5)} = 0.92857 \\
cv &= 2.7857 \times 10^5 \text{kps} \\
v_G - cv &= 111430 \text{kps} \approx (0.37143) c
\end{align*}
\]

For the classroom:

*Use Einstein’s model for the baseball problem. What is the difference between the two models in this case? Could this difference be detected by a radar unit? Could it be detected by the most sophisticated equipment currently available?*
### MDTP Written Response Items

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Level 3 and beyond

Your car, which you bought new, now has 50K miles on it. You have had all regular service performed, have not yet replaced anything, and the car is in perfect condition. However, you are noticing odometer readings inconsistent with familiar trips.

Some working assumptions:

Gas mileage is stable in modern vehicles free of mechanical defects if proper gas grade is used.
Driving habits have not changed.
Tires typically lose $\frac{1}{16}$ of tread every 10K.

Mathematical tools:

Geometry (assume outside diameter of new tire is 24")
Distance-Rate-Time

When the car was new, the odometer registered the distance from home to work as 12.6 miles. What does it now show the distance to be?

$$\frac{24 - 2 \left( \frac{5}{16} \right)}{24} \approx 0.97396$$
$$\frac{12.6}{0.97396} \approx 12.9$$

If the speedometer reads 65mph, how fast are you going?

$$65 \times (0.97396) \approx 63.307$$

You realize you need new tires but the size you need, 225/55R16, is not available. The dealer says 235/60R16 will fit your car. How will this affect the odometer and speedometer readings?

Bonus Question: Your car displays average fuel consumption (mpg) as you drive. Is this reading dependent on tire size?
What is an angle?

A circular table top 3 meters in diameter is to be inscribed with a triangular tile whose angles are given. There is a mark where one of the vertices is to be located. You must mark the other two vertices so that the tile can be custom manufactured for this table. You have no tools, but the edge of the table is calibrated in millimeters, from zero, at the given mark, all the way around the circumference.

Strategy: Express angles in radian measure and use Euclid.
Central angle from millimeter mark \( m_j \) is

\[
\begin{align*}
t_j &= \frac{m_j}{r} = \frac{1}{1500} m_j \\
m_0 &= 0
\end{align*}
\]

Euclid says:

\[
\begin{align*}
\alpha_0 &= \frac{1}{2} (t_2 - t_1) \\
\alpha_1 &= \frac{1}{2} (2\pi - t_2) \\
\alpha_2 &= \frac{1}{2} t_1
\end{align*}
\]

Given \( \alpha_0, \alpha_1, \alpha_2 \):

\[
\begin{align*}
m_1 &= 3000 \alpha_2 \\
m_2 &= 3000 (\alpha_0 + \alpha_2)
\end{align*}
\]
Question: Suppose you are given $\alpha_1 = 30^\circ$ and $\alpha_2 = 50^\circ$. 