

NCTM National Conference, 2017

Arithmetic & Algebra

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Related Common Core Standards

- ❖ First instance of “variable”:

CCSS.MATH.CONTENT.1.OA.A.1

Use addition and subtraction within 20 to solve word problems involving situations of adding to, taking from, putting together, taking apart, and comparing, with unknowns in all positions, e.g., by using objects, drawings, and *equations with a symbol for the unknown number* to represent the problem.¹

Related Common Core Standards

First formal instance of variable:

CCSS.MATH.CONTENT.3.OA.D.8

Solve two-step word problems using the four operations.

Represent these problems *using equations with a letter standing for the unknown quantity*. Assess the reasonableness of answers using mental computation and estimation strategies including rounding.

Related Common Core Standards

First formal solving of equations:

CCSS.MATH.CONTENT.6.EE.B.5

Understand solving an equation or inequality as a process of answering a question: which values from a specified set, if any, make the equation or inequality true? Use substitution to determine whether a given number in a specified set makes an equation or inequality true.

CCSS.MATH.CONTENT.6.EE.B.6

Use variables to represent numbers and write expressions when solving a real-world or mathematical problem; understand that a variable can represent an unknown number, or, depending on the purpose at hand, any number in a specified set.

CCSS.MATH.CONTENT.6.EE.B.7

Solve real-world and mathematical problems by writing and solving equations of the form $x + p = q$ and $px = q$ for cases in which p , q and x are all nonnegative rational numbers.

Connecting Arithmetic & Algebra

Institute of the Faculties

OF THE NORMAL SCHOOLS,

Held at Oshkosh, December 17-21, 1900.

CONDUCTOR,

HON. L. D. HARVEY,

State Superintendent.

Madison, April, 1901.

PURPOSE AND PLAN IN TEACHING ALGEBRA.

JOSEPH V. COLLINS, Stevens Point.

Algebra unlike most other branches in the elementary course, stands by itself in two or three important particulars. For one thing, in our teaching it is not very closely connected with either arithmetic or geometry, and not being closely connected with them is therefore not very closely related with any other study. Then again, if most teachers were to be asked what is the good of the study of algebra, they would have difficulty in giving an answer satisfactory even to themselves.

Connecting Arithmetic and Algebra

Are the story problems of elementary school and those we find in algebra class related and can they be used to connect algebra and arithmetic?

Let's solve with and without algebra...

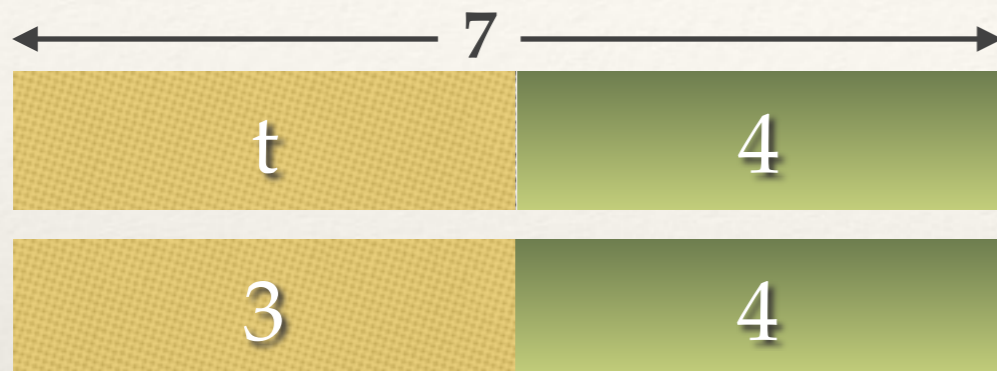
1. J. had some model trucks. Today he bought four more. Now he has seven trucks. How many trucks did he have?

How would we solve this in elementary school?

How would we solve this in algebra class?

Note: This problem is an Add To, Start Unknown (Table 1, Appendix A, Common Core Math Standards), which is the most difficult of the Add To types for students to solve.

Various Solutions



If J. ended up with seven trucks after buying four new trucks, then before he bought the new trucks, he had $7 - 4 = 3$ trucks.

# of trucks he had	# trucks he has now	totals 7?
4	$4+4 = 8$	no
2	$2+4 = 6$	no
3	$3+4 = 7$	yes!

Let t be the # of trucks J. had.
Then, $t + 4 = 7$
and $(t + 4) - 4 = 7 - 4$
 $t = 3$.

How do we want to characterize our various solution methods?

One proposal:

Diagrammatic

Verbal

Guess and Check

Algebraic

Let's solve with and without algebra...

2. J. bought three packs of balloons. He opened them and counted 12 balloons. How many balloons are in a pack?

Record your work on scratch as follows:

Problem:	
Diagrammatic, Numeric, Verbal (or other!) Solution Method	Algebraic Solution Method
Connections between the solution methods:	

Let's solve with and without algebra...

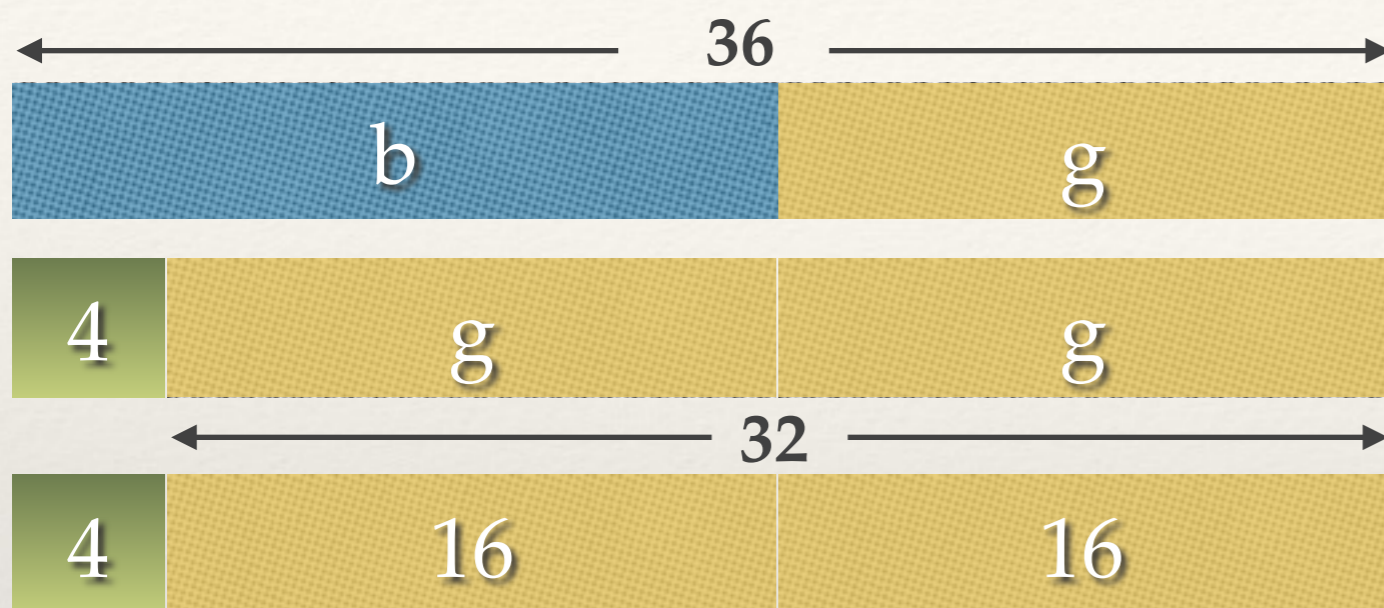
3. J. has 4 packages of balloons and five single balloons. In all he has 21 balloons. How many balloons are in a package?

Let's solve with and without algebra...

4. There are 36 children in a class. There are 4 more boys than girls. How many boys are in the class, and how many girls?

Note: This problem is something of a watershed in the United States, almost always being approached algebraically.

Various Solutions:



Take away four boys. This leaves $36 - 4 = 32$ students, and an equal number of boys and girls. Therefore there are $32 / 2 = 16$ girls.

# of girls	# of boys	# of students	totals 36?
4	$4+4=8$	$4+8=12$	no
10	$10+4=14$	$10+14=24$	no
16	$16+4=20$	$16+20=36$	yes!

Let b be the # of boys in the class, and let g be the # of girls.

Then, $b + g = 36$ and $b = g + 4$.

Solving we have:

$$b + g = (g+4) + g = 2g + 4.$$

$$\rightarrow 2g + 4 = 36, \rightarrow 2g = 36 - 4 = 32,$$

$$\rightarrow g = 2g/2 = 32/2 = 16.$$

Let's solve with and without algebra...

5. If a bar of soap balances $\frac{3}{4}$ of a bar of soap and $\frac{3}{4}$ of a pound, how much does the bar of soap weigh?

Various Solutions:

Since there is a full bar of soap on one side of the balance, and $3/4$ bar of soap on the other, we may remove $3/4$ of a bar of soap from each side of the balance.

The objects remaining on each side will still balance.

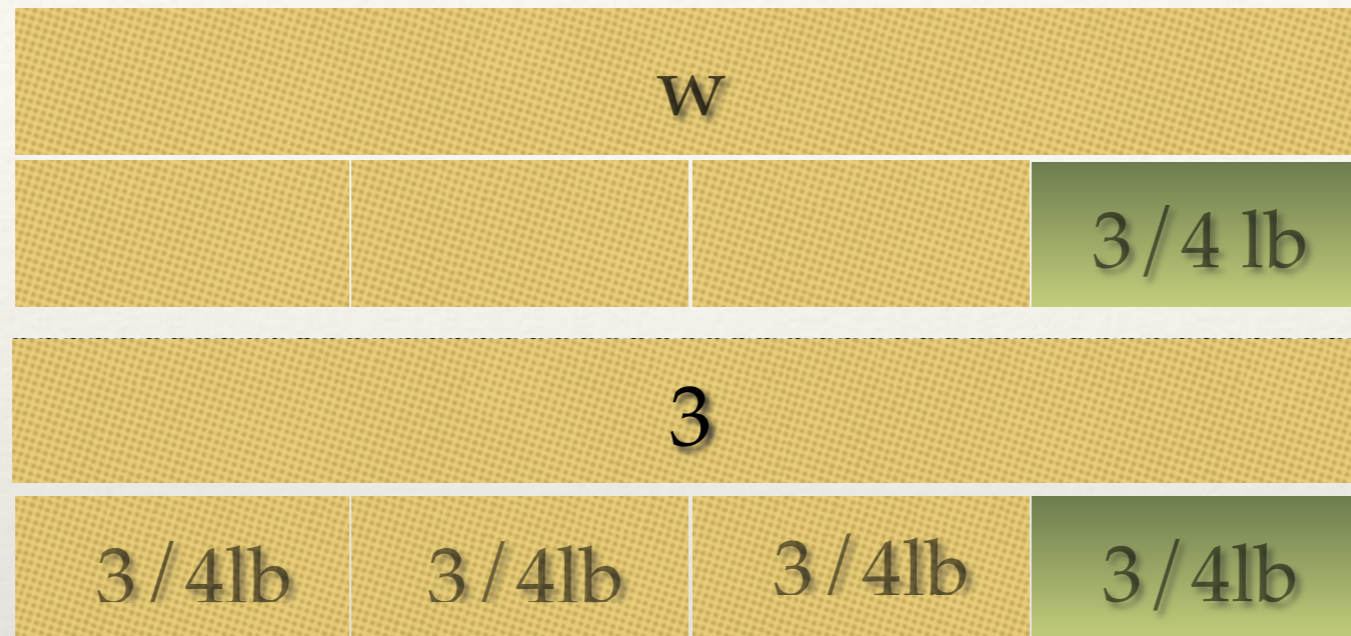
Thus $1/4$ bar of soap weighs $3/4$ pounds. A full bar of soap is $4 \times 1/4$ ths, so it weighs $4 \times 3/4 = 3$ pounds.

Let S be the weight of a bar of soap, in pounds. The weight of one bar of soap is S . The weight of $3/4$ bar of soap and $3/4$ pounds is $3/4 S + 3/4$.

Hence we have $S = 3/4 S + 3/4$.

Solving, we have $S - 3/4 S = 1/4 S = 3/4$,
 $\rightarrow S = 4 \times 1/4 S = 4 \times 3/4 = 3$.

Various Solutions:



weight of one bar, in lbs.	weight of 3/4 bar and 3/4 lbs.	balanced?
1	$3/4 \times (1\text{lb}) + 3/4 \text{ lb} = 1.5 \text{ lbs}$	no
1/2 lb	$3/4 \times (1/2 \text{ lb}) + 3/4 \text{ lb} = 9/8 \text{ lbs}$	no

Note: Creating the second column on the guess and check table is non-trivial for many students.

For each problem, please check the two solution methods you believe students would find most accessible.

Problem #	K-8			6-8		
	Diagrammatic	Verbal	Guess & Check	Algebraic	D → A	GC → A
1						
2						
3						
4						
5						
6						
7						

Let's solve with and without algebra...

6. A man wants to share his coins equally among some friends. If he gives each friend 6 coins, he would have 4 coins left over. If he gives each friend 7 coins, he would be 5 coins short. With how many friends does he share?

Various Solutions:



# of friends	total coins if sharing 6 each	total coins if sharing 7 each	same # of coins?
3	$6 \times 3 + 4 = 22$	$7 \times 3 - 5 = 16$	no
7	$6 \times 7 + 4 = 48$	$7 \times 7 - 5 = 44$	no
9	$6 \times 9 + 4 = 58$	$7 \times 9 - 5 = 58$	yes!

Various Solutions:

How does working with the diagrammatic model become more challenging when the student must model a groups of b objects where a is unknown?

Consider the challenge of the diagrammatic model when solving the following problem:

H. and J. are gaining weight for football. H. weighs 205 pounds and is gaining 2 pounds per week. J. weighs 195 pounds and is gaining 3 pounds per week. When will they weigh the same?

Various Solutions:

# of friends	total coins if sharing 6 each	total coins if sharing 7 each	same # of coins?
3	$6 \times 3 + 4 = 22$	$7 \times 3 - 5 = 16$	no
7	$6 \times 7 + 4 = 48$	$7 \times 7 - 5 = 44$	no
9	$6 \times 9 + 4 = 58$	$7 \times 9 - 5 = 58$	yes!
f	$6f + 4$	$7f - 5$	$6f + 4 = 7f - 5$

$$6f - 6f + 4 = 7f - 6f - 5$$

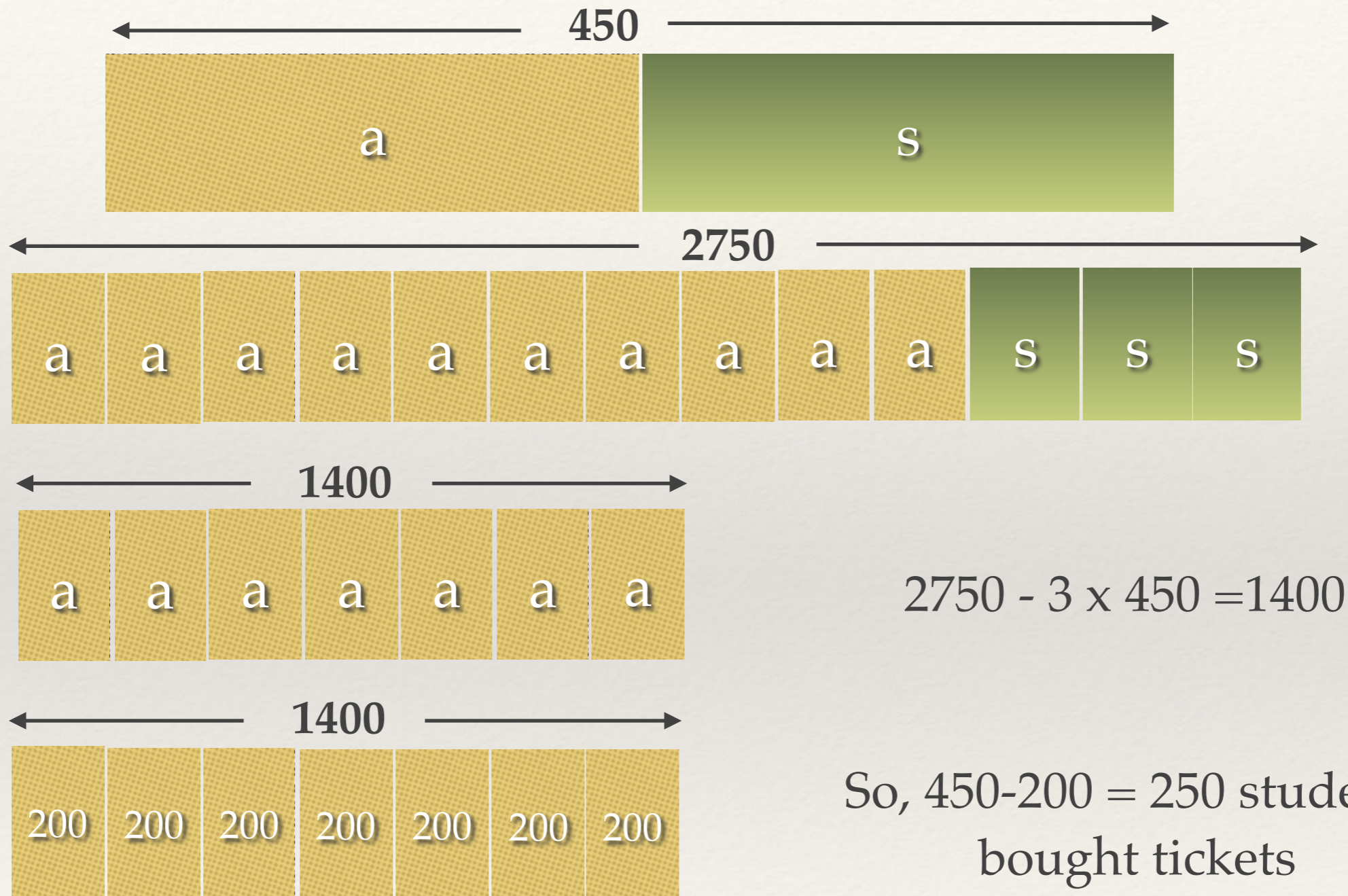
$$4 + 5 = 1f - 5 + 5$$

$$9 = 1f$$

Let's solve with and without algebra...

7. Tickets for the class show are \$3 for students, and \$10 for adults. The auditorium holds 450. The show was sold out, and the class raised \$2750 in ticket sales. How many students bought tickets?

Various Solutions:



Note: In this problem, the bar model becomes more challenging to both draw and to conceptualize as students deal with modeling a groups of b where a is unknown.

Various Solutions:

The class sold 450 tickets.

If all the ticket had been for adults, total sales would have been \$4500.

Instead, the sales were \$2750, which is $\$4500 - \$2750 = \$1750$ less.

Since each student ticket brings in $\$10 - \$3 = \$7$ less than an adult ticket, there must have been $1750/7 = 250$ student tickets sold.

Note: This is called “*method of false position*”. Note how this non-algebraic method requires more ingenuity on this problem compared to the algebraic method.

Let s be the number of student tickets sold, and let a be the number of adult tickets sold.

Then $a + s = 450$ and

$$10a + 3s = 2750.$$

$$a + s = 450 \quad a = 450 - s$$

$$10(450 - s) + 3s = 2750$$

$$4500 - 10s + 3s = 2750$$

$$4500 - 7s = 2750$$

$$4500 - 4500 - 7s = 2750 - 4500$$

$$-7s = -1750$$

So, $s = -1750 / -7 = 250$ student tickets

Various Solutions:

# of student tickets	income from student tickets	income from adult tickets	total income	totals \$2750?
20	$\$3 \times 20 = \60	$\$10 \times (450 - 20) =$ $\$10 \times 430 = \4300	$\$60 + \$4300 = \$4360$	no
100	$\$3 \times 100 = \300	$\$10 \times (450 - 100) =$ $\$10 \times 350 = \3500	$\$300 + \$3500 =$ $\$3800$	no
s	$\$3 \times s = 3s$	$\$10 \times (450 - s) =$ $10(450 - s)$	$3s + 10(450 - s) = 2750$	

$$3s + 10(450 - s) = 2750$$

$$3s + 4500 - 10s = 2750$$

$$4500 - 7s = 2750$$

$$7s = 1750$$

$$s = 250$$

Check the solution method(s) you believe particularly aid students in accomplishing the indicated task.

TASKS	SOLUTION METHODS					
	Diagrammatic	Verbal	Guess & Check	Algebraic	D → A	GC → A
Identify the quantity asked for						
Identify the other quantities involved						
Identify the relationships between the quantities						
Solve the problem						
Understand the solution						

Connecting Arithmetic and Algebra

We believe the story problems of elementary school and those we find in algebra class *ARE* related and *CAN* be used as one of the connections between algebra and arithmetic.

Other Avenues for Connecting Arithmetic and Algebra

Via our notation for numbers:

$$156 = 1 \times 10^2 + 5 \times 10^1 + 6 \times 10^0$$

$$\begin{aligned} 156 &= 1x^2 + 5x^1 + 6x^0 \\ &= x^2 + 5x + 6 \end{aligned}$$

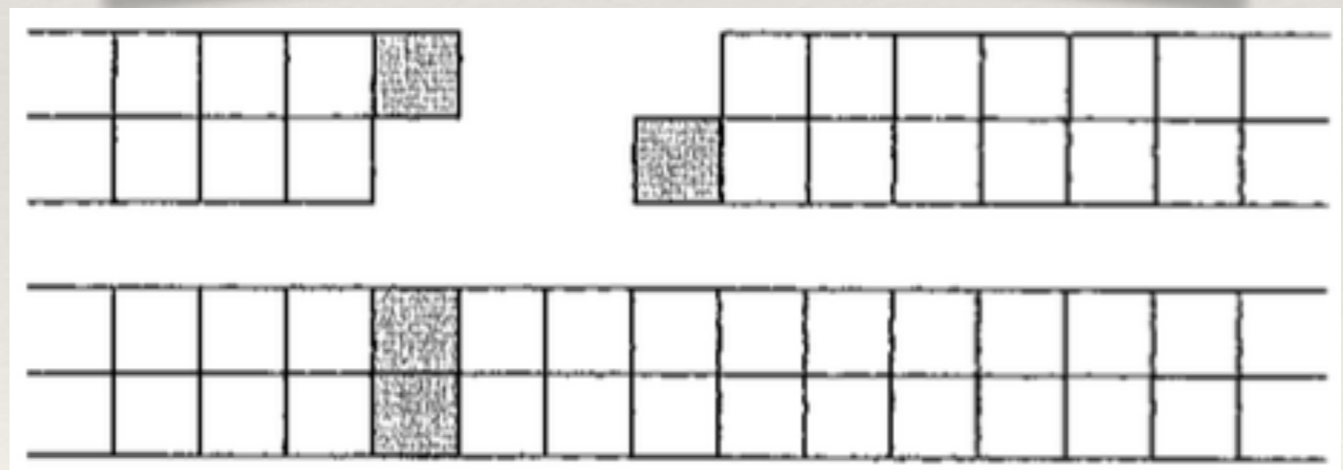
when $x = 10$

Via Proof:

Ask students to justify general claims about numbers.

Example:

The sum of two odds is even.



Russell, Schifter & Bastable:

<http://homepages.wmich.edu/~grantt/Developing%20Mathematical%20Arguments.pdf>

Other Avenues for Connecting Arithmetic and Algebra

Via the four operations:

$$\begin{array}{r} 12 \\ \times 13 \\ \hline 36 \\ 120 \\ \hline 156 \end{array}$$

$$\begin{array}{r} 10 + 2 \\ \times 10 + 3 \\ \hline 6 \\ 30 \\ 20 \\ \hline 100 \\ 156 \end{array}$$

$$\begin{array}{r} x + 2 \\ \times x + 3 \\ \hline 6 \\ 3x \\ 2x \\ \hline x^2 \\ x^2 + 5x + 6 \end{array}$$

Questions

- ❖ For which (if any) problems was the algebraic solution method more advantageous? If any, how?
- ❖ Which non-algebraic methods (if any) prepare students for the algebraic solution method? If any, how?
- ❖ What (if anything) in this talk helped you think about using connections between arithmetic and algebra in your classroom?

Contact and resources

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- ❖ roger.howe@yale.edu
- ❖ Google: UCLA Curtis Center and click on “Latest Resources” for the slides from this talk & Roger’s paper “Arithmetic to Algebra”

From Arithmetic to Algebra

Roger Howe

Abstract: In the United States, many students have trouble learning algebra. Also, many students form the impression that algebra has little or nothing to do with arithmetic. This paper suggests that teaching algebra through analysis and discussion of word problems, including comparison of arithmetic and algebraic solutions, might help students better understand what algebra is about. It would also provide a context