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Themes of this issue:

Data Science in the Mathematics Classroom
and
Focusing on Student Assets to Build Mathematical Competence

The Concrete-Representational-Abstract Teaching Method

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Concrete
Representational
Abstract

The UC Los Angeles Curtis Center for Mathematics and Teaching has for over 15 years employed a pedagogical strategy called the Concrete-Representational-Abstract (CRA) approach. Grounded in sixty-years of research, this effective strategy:

- ✔ *begins* with the use of a concrete model such as algebra tiles to represent new mathematics,
- ✔ *follows* with the use of a representation of the concrete model, most often a diagram, and
- ✔ *culminates* with the use of a traditional abstract representation of the new mathematics, such as expressions and equations.

Examples of concrete models are Unifix cubes for teaching addition or cups and tile spacers to teach equation-solving. Once at the abstract stage, students extend their learning beyond the limits of the concrete model. This article summarizes some of the CRA strategy research and then provides a sample CRA lesson.

CRA Research

In a 2005 study, Bradley Witzel, award-winning teacher and researcher, found that using CRA promotes information retrieval since instructors have presented the information in various ways: visual, auditory, kinesthetic, and tactile (2005). This variety provides students with multiple means to access mathematics. Along with other colleagues, Witzel conducted another CRA research study in 2008, where he further states that instructors who taught using CRA strategies helped students make personalized, concrete, and meaningful connections with mathematics (Witzel, Riccomini, and Schneider 2008).

In the spring of 2009, Margaret Flores attempted to research effective CRA models for teaching subtraction with regrouping to elementary students. When Flores could not find research studies substantiating the effectiveness of using CRA to teach this concept, she decided to conduct her own. Flores created lessons using base-10 blocks, and another teacher taught these lessons to a small group of students.

All the students in the study had previously failed mathematics and struggled to learn subtraction with regrouping. Based on her analysis, Flores reported that through CRA, “the students physically demonstrated their understanding of the regrouping procedure rather than memorizing steps.” Based upon teacher interviews, Flores reported that “students’ regrouping performance on district-mandated mathematics benchmark testing increased and that [teachers] would recommend the strategy to other teachers.” Students reported that “they liked the strategy, it made subtraction easier, and they would recommend it to other students.” Flores concluded that “the CRA instructional sequence provided the students with a scaffold from conceptual understanding to procedural knowledge in which the students became fluent” (Flores 2009).

In a 2013 study, Kira J. Carbonneau, Scott C. Marley, and James P. Selig (2013) examined and statistically analyzed 55 studies on the effectiveness of teaching mathematics concepts using concrete models, comparing it to the teaching of the same concept without concrete models. In their study, the researchers found that the success of CRA is strongly influenced by the level of guidance a teacher offers students when first learning how to use concrete models. If a teacher gives concrete models to students with minimal to no introduction, student learning does not benefit much from using CRA.

There are several key takeaways from the research regarding the CRA teaching strategy model:

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- This strategy can aid students with the retrieval of information.
- CRA lessons allow for multiple means of accessing the mathematics concepts: visual, auditory, kinesthetic, and tactile.
- The CRA model provides scaffolding for students.
- CRA strategies can increase student assessment performance.
- The effectiveness of the CRA teaching model may be influenced by how much time a teacher takes to introduce how to use concrete objects when first learning the concept.
- Students reported liking the CRA method of learning and would recommend it.

CRA Model: A Sample Lesson

We now examine a UC Los Angeles Curtis Center lesson utilizing CRA: "Using Cups and Tile Spacers to Teach Solving Single Variable Equations."

Materials Needed

The lesson uses *tile spacers* as a concrete model of positive and negative numbers. These are small pieces of rubber shaped like the plus sign (+) and may be purchased in large quantities for low cost at most hardware stores. Using tile spacers, we represent +1 with one tile spacer since it has the shape of the positive sign (+) To represent –1, we cut two opposing prongs from a tile spacer to achieve the shape of a negative sign (–) (***Figure 1***).

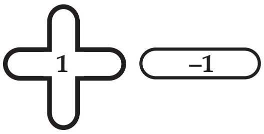


Figure 1

The lesson also hides the tile spacers under the cups to concretely represent an unknown number. We suggest three-ounce paper cups, but any size will work.

The Lesson

Step 1: Defining –1 and Introducing Multiple Representations of 0

First, we (the teachers) introduce the (+) tile spacer to students as a representation of +1 and the cut tile spacer (–) as a representation of (–1). Next, we provide students with a definition of –1: *It is the number whose sum with* +1 *is* 0. In other words, –1 *is the number such that* (–1) + (+1) = 0. This definition implies that the numeric value represented by putting together a (+) tile spacer and a (–) tile spacer is 0 (***Figure 2***).

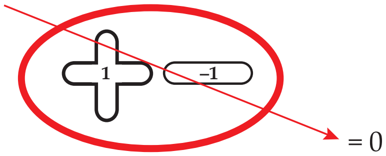


Figure 2

We also demonstrate to students that putting together any equal amount of (+) and (–) tile spacers will result in a representation of 0 and is called a "zero field" (***Figure 3***).

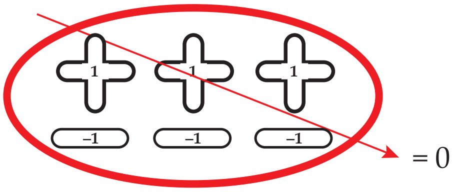


Figure 3

Step 2: What is Under the Cup? Puzzle 1

In this step, we discretely hide three (+) tile spacers under each of two upside-down paper cups and place them together with three (+) tile spacers (see the left side of the equal sign in ***Figure 4***). We then write an equal sign and position nine tile spacers on the opposite side, as represented in ***Figure 4***.

Figure 4

We then tell students they are going to solve a puzzle. We say that there are an equal number of (+) tile spacers hidden under each of the two cups and ask: *What is the numeric value represented by the (+) tile spacers under each cup?*

We give students a few minutes to think and then ask them to turn to a partner and discuss their process and answer. After a few minutes of discussion, we ask students (one from each pair) to report by sharing their discussion with the whole group. Students often state that their answer is 3 and explain that they disregarded the 3 (+) tile spacers on the left along with three of the (+) tile spacers on the right. This discussion allows them to focus on the two cups on the left and six of the (+) tile spacers on the right. They then concluded that the numeric value represented by the tile spacers hidden under each cup was 3.

Step 3: Representing Our Thinking

Together with the class, students carefully represent this solution process with the concrete models on their desks (C), drawing diagrammatic representations of the concrete model (R) on their papers, and finally, write the traditional abstract mathematical equations (A). We show students a chart to keep track of their thinking and ask them to create a copy of the chart on their papers, labeling three columns: *Diagram, Equation,* and *Check*.

In the Diagram column (***Figure 5***), we sketch the concrete model, using a circle to represent cups hiding tile spacers. In the Equation column, we represent the concrete model with an equation and remind students that "*c*" represents *the numeric value represented by the tile spacers hidden under the cup*, not the cup itself. We will use the Check column later.

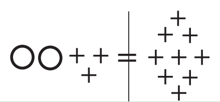
Diagram	Equation	Check
	$c + c + 3 = +9$	

Figure 5

As a class, we relate the discarding of the three (+) tile spacers on the left along with three of the (+) tile spacers on the right to physically taking away three (+) tile spacers on the left and three (+) tile spacers on the right. In addition, we represent the taking away in our class diagram and equation and the students copy the information into their charts. (***Figure 6***):

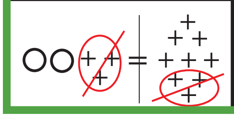
Diagram	Equation	Check
	$2c + 3 - 3 =$ $+ 9 - 3$	

Figure 6

After students cross out the three (+) tile spacers on each side of the diagram, they then sketch the result. We ask: *What do you see on the right side of the equal sign and the left side of the equal sign? Represent what is remaining with an equation in the Equation column* (***Figure 7***).

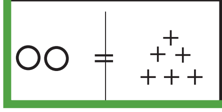
Diagram	Equation	Check
	$2c = +6$	

Figure 7

We now ask students to think about what mathematical operation is related to concluding that the value of the tile spacers hidden under each cup is 3, given that there are two cups on the left and six (+) tile spacers on the right. We physically separate the two cups and, knowing the same number of tile spacers hide under each cup, we match one tile spacer with the first cup, the next tile spacer with the second cup, and so on until we have matched an equal number of tile spacers with each cup (***Figure 8***).

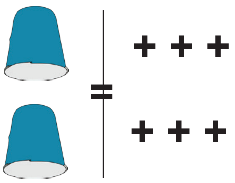


Figure 8

At this point, we ask students what mathematical operation they engaged in when they equally shared the tile spacers among the cups. Often, students generate the idea that they are dividing by 2. If not, we ask guiding questions to facilitate this conclusion. We represent this division in our diagrams and equation (***Figure 9***):

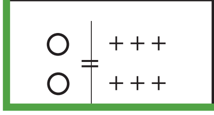
Diagram	Equation	Check
	$\frac{2c}{2} = \frac{+6}{2}$	

Figure 9

Now it is time for the grand reveal! We ask students to remove one cup and its corresponding three (+) tile spacers and represent the numeric value represented by the tile spacers hidden under the remaining cup with a diagram and equation (***Figure 10***).

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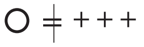
Diagram	Equation	Check
	$1c \;=\; +3$	

Figure 10

Finally, we complete the Check column where we substitute +3 for "*c*" and check that the resulting equation is true (***Figure 11***):

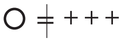
Diagram	Equation	Check
	$1c \;=\; +3$	$2(3) + 3 \overset{?}{=} 9$ $6 + 3 \overset{?}{=} 9$ $9 = 9$ √

Figure 11

Step 4: Puzzle 2

We now present the students with a second puzzle that involves a negative number. We discretely hide five (+) tile spacers under each of the three upside-down paper cups and place them together with seven (–) tile spacers. We then write an equal sign and position eight (+) tile spacers on the opposite side, represented in ***Figure 12***:

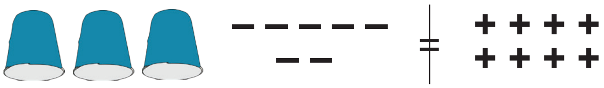


Figure 12

We tell students that we hid an equal number of either (+) tile or (–) tile spacers under each cup and ask them: *What is the numeric value represented by the tile spacers hidden under each cup?*

In this case, students will notice that one cannot disregard the seven (–) tile spacers on the left. But we prompt them to consider eliminating them by asking: *What tile spacers could we add to the left side so that the total value represented by the tile spacers on that side is 0?* Students demonstrate the answer with their own cups and tile spacers model. After students work for a few minutes, we ask them to turn to a partner and discuss their process and answer. After sharing, a student from each pair reports their discussion to the whole group. Often students express that their solution was to add seven (+) tile spacers to the left side of their model to create a group of tile spacers that represent 0. It might be necessary to ask students whether adding seven (+) tile spacers on the left changes the truth of the equation to prompt discussion around the need to add seven (+) tile spacers to the right side of their model. Again, we ask students to represent their actions with their own cups and tile spacers model and with a diagram and equation (***Figure 13***):

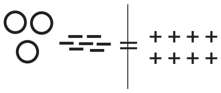
Diagram	Equation	Check
	$3c + (-7) = 8$	

Figure 13

Next, students represent placing seven (+) tile spacers on both sides of the equal sign and indicate that the value of seven (+) tile spacers and seven (–) tile spacers is 0 (***Figure 14***).

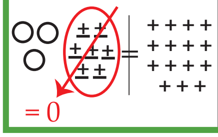
Diagram	Equation	Check
	$3c + (-7) + 7$ $= 8 + 7$	

Figure 14

Next, we ask students to remove this zero field from their diagram and represent the result with a diagram and an equation (***Figure 15***):

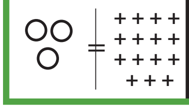
Diagram	Equation	Check
	$3c = +15$	

Figure 15

We now ask students to work with their physical cups and tile spacers to find the numeric value represented by the tile spacers hidden under each cup, independently or with one or two of their peers. Like the first puzzle, students should physically separate the three cups and, knowing the same number and same type of tile spacers hides under each cup, they match one tile spacer with the first cup, the next tile spacer with the second cup, and so on until they have equally shared the tile spacers among the three cups, as shown in ***Figure 16***. After sufficient time, we observe students' progress with this and ask student reporters to share their approach with the whole class.

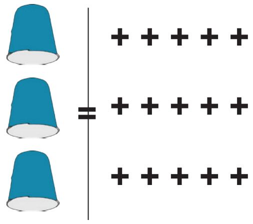


Figure 16

We now ask students to represent their work with the cups and tile spacers with diagrams and equations. First, they represent the equal sharing between the three cups and also divided by 3 on each side of the equation (***Figure 17***):

Diagram	Equation	Check
○ +++++ ○ = +++++ ○ +++++	$\frac{3c}{3} = \frac{+15}{3}$	

Figure 17

Next, students remove two of the cups and their corresponding tile spacers and represent the value of the tile spacers hidden under one cup with a diagram and an equation (***Figure 18***):

Diagram	Equation	Check
○ = +++++	$1c = +5$	

Figure 18

Now it is time for another grand reveal! We ask a student to lift one of the cups in the tile spacers and cups model we built to reveal the five tile spacers hidden beneath. Finally, we ask students to complete the Check column by substituting +5 for *c* in the original equation (***Figure 19***):

Diagram	Equation	Check
○ = +++++	$1c = +5$	$3(5) + (-7) \overset{?}{=} 8$ $15 + (-7) \overset{?}{=} 8$ $8 = 8$ √

Figure 19

Next Steps: Puzzle 3

Next, we engage students with a cups and tile spacers puzzle representing the equation $2c + 9 = +5$, which has a negative solution. Students should work on this puzzle on their own or with one or two peers to build their concrete model (C), represent it with diagrams (R), and create abstract equations (A).

While students work, we observe their process, ask appropriate guiding questions, and note those with approaches we would like to highlight with the whole class. After sufficient time, we ask these students to share their approaches with the whole class. ***Figure 20*** is an example of a potential student solution.

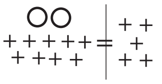
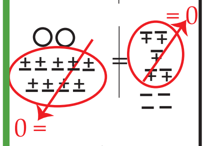
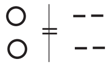
Diagram	Equation	Check
	$2c + 9 = 5$	
	$2c + 9 - 9$ $= 5 - 9$	
	$2c = -4$	
	$\frac{2c}{2} = \frac{-4}{2}$	
	$c = -2$	
		$2(-2) + 9 \overset{?}{=} 5$
		$-4 + 9 \overset{?}{=} 5$
		$5 = 5$ √

Figure 20

We assign several similar puzzles for homework and ask students to represent their solution process using the Diagram, Equation, and Check organizer. Suggested puzzles include:

- ✓ $5c - 2 = -7$
- ✓ $3 + 4c = -2$
- ✓ $1 = 3c - 5$

Suggested puzzles for the following day include:

✓ $2c + 1 = 12$	For this puzzle, cut a (+) tile spacer in half and put one-half in each cup.
✓ $-2 = 4c - 3$	For this puzzle, cut a (+) tile spacer in fourths and put one-fourth in each cup.

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- ✓ $-8 = -3 + 2c$
- ✓ $4c - 2 = -4$
- ✓ $5 = 4c - 10$

We encourage students to work independently or with one or two peers. While they work, we observe student progress, ask guiding or extending questions, and note students with approaches we would like to highlight to the whole class. At several junctures, we convene the whole class and ask these students to share their approach with the whole class.

Conclusion

This particular CRA instructional sequence aims to provide a low floor, sense-making introduction to solving equations and support student fluency in solving equations. Note that most concrete models have limitations. For instance, the Cups and Tile Spacers model should not be used to represent equations such as $-2c = 10$ since there is no such thing as a *negative cup*. To introduce a negative cup means to inject a senseless concept into a model meant to encourage sense-making.

Once their conceptual understanding develops, students may leave the concrete model behind and generalize their equation-solving approach to equations such as $-2c = 10$ or $2\pi r = 10$. One strength of this model is that it helps students troubleshoot common errors during later problem solving. For example, a common student error in solving equations like $2c = 10$ is to subtract 2 from either side of the equation and write $c = 8$. In this case, we can ask them to draw a cup and tile spacers diagram to represent the equation (***Figure 21***):

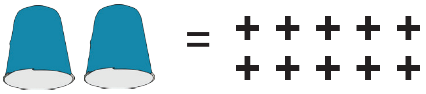

Figure 21

This drawing prompts the student to see that the natural first move is to divide the tile spacers between the two cups, which we represent algebraically by dividing both sides of the equation by 2. Similarly, when a student begins to solve $2c + 2 = 10$ by dividing $2c$ and 10 by 2 to arrive at $c + 2 = 5$, we can ask them to draw a cup and tile spacers diagram to represent that equation (***Figure 22***):

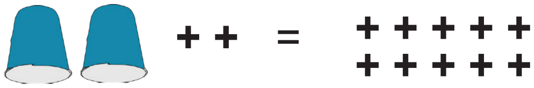

Figure 22

This diagram prompts the student to see that the natural first move is to take away two (+) tile spacers from each side, which we represent algebraically by subtracting +2 from both sides of the equation. Thus, the concrete model can build conceptual understanding and support sense-making while building fluency.

If you want to read more about the Concrete-Representational-Abstract teaching method, see the resources below.

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Resources

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