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Engaging with Text in Mathematics

ACADEMIC LANGUAGE + SYMBOLISM + VISUAL IMAGES

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Introduction

The Instructional Leadership Framework adopted by New York City public schools provides us with a view of academic language as *"a set of skills and competencies that enable communication, spoken and written, in increasingly diverse ways and with an increasingly diverse audience" (Lesaux, Galloway & Marietta, 2016)*. In this framework, academic language is developed in service of rich disciplinary content. Therefore, it calls on all members of the school community to share the responsibility to work together in order to provide opportunities for all linguistically and culturally diverse learners to develop the increasingly sophisticated language skills necessary to succeed in academic settings and beyond.

The framework identified three key shifts for school leaders to create a supportive environment and set high expectations for all students (i.e., strengthen core instruction, know every student well, and use inclusive curriculum). The first shift, strengthening core instruction, emphasizes the need for content teachers to develop the academic language of their discipline. Central to strengthening core instruction is a set of high-impact instructional practices:

It is important that math teachers support students in developing the academic language of the discipline.

- *Work with a variety of texts[1] that feature big ideas and rich content.*
- *Talk/discuss[2] to build language and knowledge.*
- *Use extended writing[3] as a platform to build language and knowledge.*
- *Study a small set of high-utility vocabulary words needed to master language and content.*

Through this document guide, we strive to offer guidance on how to realize these high-impact instructional practices in mathematics. In our experience, math teachers may encounter the most difficulty with *"engaging students with mathematical text."* For this reason, this document guide is focused on explicit support to promote "engaging students with mathematical text."

Part I of this guide offers an account of the current state of affairs and provides detailed characteristics of academic language in mathematics while making a case for the need for explicit teaching of academic language development that is specific to the discipline.

1 Interpreted broadly to be printed texts, digital texts, and visual texts, including phenomenon-based inquiries and modeling.

2 Engage in mathematical discourse and communication.

3 Including visual depiction of mathematical content in graph and figures.

Part II provides an integrated approach of strategies, routines, scaffolds, and analysis of the language demands to support students' engagement with mathematical texts.

Part III offers a collection of mathematical texts for various grade levels around important mathematical ideas. It is important to note that the ideas, recommendations, and resources offered in this document guide do not reflect an exhausted list but rather a beginning. This document guide can be viewed as a starting point that can be strengthened by contributions from math teachers from all backgrounds and experiences.

Part I: Understanding the Complexity of Academic Language in Mathematics

Access and Equity

Mathematics is itself a language—a language we use to do much more than communicate. Still, to talk and think about mathematics, we need a broader academic language, one that encompasses more than mathematics. We need a rich academic language that enables us to talk about the scope of mathematics: its symbols, structures, references, genres, and assumptions in the speaker-listener and writer-reader implied contracts. Ultimately, the language and specialized representational tools of mathematics are thinking tools as well as communication tools. The academic language needed to read, write, and talk about mathematics goes beyond the glossary of mathematical terms and includes many of the same words and syntax used in science and argumentation. Short words, like prepositions and conjunctions, are used in precise and critical ways. Learning this academic language is part of learning to think and talk about mathematics.

Unless we meet this language development challenge, students will not have access to learning mathematics itself. Too many students are denied the opportunity to learn mathematics because they cannot comprehend or produce the academic language in which mathematics is embedded. In mathematics, the reading and writing challenge can be great enough to block mathematics learning. Readers must deploy their knowledge of mathematical terms and notations deeply rooted in a peculiar dialect of academic English to make sense of text at the most basic level. Often, diagrams in the text help students to understand the content. The logical consistency of mathematics can make comprehension easier—even while the logical complexity makes it more difficult.

The State of Academic Language Development in Mathematics

As important as language development is for mathematical thinking, it is systematically neglected in American curricula and teaching traditions *(Greenleaf & Valencia, 2017).* Attention to language is usually limited to definitions of terms. This neglect of academic language development costs students the opportunity to comprehend grade-level mathematics each year. The mastery of ordinary language tends to be an assumption of mathematical exposition. Too much of mathematical exposition in school focuses on formal development. It takes for granted that the necessary intuition is in place. For example, the lack of development of intuitive understanding of base ten concepts hides under the never-explained and poorly understood code words "place value."

The price paid for weak academic language development accumulates year by year, so that by the time students reach the language-intense demands of algebra, they are linguistically ill-equipped to make sense of what is being said or read, nor can they adequately express their own mathematical thinking. And ultimately, their thinking is stunted by the lack of linguistic and representational tools that should have been developed in every grade level of mathematics instruction.

The price paid for weak academic language development accumulates year by year, until ultimately, thinking is stunted by the lack of linguistic and representational tools.

In addition, available materials neglect many essential ingredients for language development. Students are fed text about lessons with the mathematics embedded in the metalinguistic conventions of lesson jargon. They are deprived of text that speaks directly about the mathematics in a linguistic context of age-adapted mathematical language.

To develop academic language, students need to engage in four modes of language use (i.e., listening, speaking, reading, and writing). These modes interact and support each other. Engagement in the four modes of language is essential for all genres of mathematical text.

In our experience, there is a tendency in math education to neglect language production in speaking and in writing. We need shifts in pedagogic process in mathematics that establish greater investment of time and effort in students being listened to (e.g., pair work as is typical in reading comprehension pedagogies) and in student writing, including revision and pre-writing processes, where students get ideas from reading and from other students.

Figure 1: Four Modes of Language

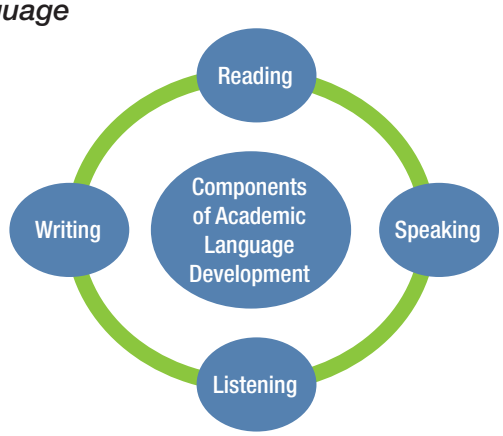

Table 1: This table can be used as a self-assessment tool to see the invisible neglect of some modes and overuse of others.

Attention to modalities of language development in mathematics instruction				
4 = almost daily | **3** = weekly | **2** = once or twice per unit | **1** = almost never				
	Listening	**Speaking**	**Reading**	**Writing**
Word problems				
Exposition				
Argument				
Procedure				

As with language development in other areas, there is valuable interdependence among the four modes.

Unique to Mathematics

Students read, write, and discuss mathematics texts[4] for at least four common purposes *(Fang, 2012, pp. 45–53)*:

- To make sense of and solve problems
- To learn mathematical concepts and systems of concepts
- To learn mathematical procedures
- To evaluate the validity of statements and arguments

Associated with these purposes are genres of mathematical text. These genres, like any genre, have conventions and features that must be learned. Unlike genres encountered in other subjects, schools invest little time and effort teaching the craft and comprehension of genres in mathematics *(Greenleaf & Valencia, 2017); (O'Halloran, 2005)*.

In our experience, the most important genres in K–12 school mathematics are:

1. Word problems
2. Explanations of mathematical concepts (i.e., expositions)
3. Proofs and arguments
4. Procedures

Each of these genres is concise and dense. The procedural instructions are common and easier to follow than to remember. Proofs and arguments in mathematics, unlike those in science, do not allow empirical evidence. Mathematical arguments feature strings of statements (propositions) that follow from prior statements or established results from prior mathematics, such as definitions, properties, and earlier arguments. The conceptual expositions can be dense and abstract. A valid mental model built from text explaining a concept is the understanding of mathematics. Finally, students must read word problems—for many, the most unpopular genre of written text ever devised.

Word Problems

Word problems are about the relationships between the numbers, not the relationships between the people. They are usually about how many or how much (i.e., quantities). Learning to comprehend word problems means learning to pay attention to how many or how much rather than to the people, their character or motives, as you might do when comprehending a story. The relevant prior knowledge to activate is prior knowledge of similar quantitative

4 Interpreted broadly to be printed texts, digital texts, visual text, including phenomenon-based inquiries and modeling.

situations. For example, if a word problem is about comparing the height of two waterfalls, the relevant prior knowledge is prior experiences comparing heights, not prior knowledge of waterfalls: “Have you ever compared your height with your mom’s?” not “Have you ever seen a waterfall?” The word problem is not about waterfalls; it is about heights. It is about how many units tall something is. It could be two buildings, two people—it does not matter. What matters is the comparison of how many units one is compared with the other.

Explanations of Mathematical Concepts

This program guide focuses on exposition/explanation, in particular learning to read and talk about mathematical explanations in expository essays aimed at grade-level audiences. Of course, the role of reading in learning mathematics is ancillary to what students do, talk about, and think. Yet it is an essential role. Talking, doing, and thinking are ephemeral except for what the students keep in the privacy of their minds. Text endures: it is available for return visits and for shared visits. Moreover, text can be a source of expertise that goes beyond the resources of the classroom. Bear in mind that mathematical text is typically rich with symbols, diagrams, graphs, tables, and other visual representations. This multisemiotic[5] nature of mathematical text reflects the nature of mathematical concepts, which are likewise rich in visual as well as symbolic and linguistic components *(Mathematical Discourse: Language, Symbolism and Visual Images, 2005, p. 10)*.

The role of reading in learning mathematics is ancillary to what students do, talk about, and think. Yet it is an essential role.

Explanations of mathematical concepts in texts are written with the assumption that the reader will be engaged in work on related problems, classroom discussions, and activities designed to build concrete intuitions that can support understanding the concept. In this sense, reading comprehension is embedded in a social context designed for learning. In order for students to build mental models of the concepts explained in the text, instruction must include a combination of concurrent experience, prior learning of the mathematical concept, and the text itself.

Mathematical explanations typically depend on previously learned concepts in detailed and precise ways, while the misconceptions that students bring to their reading can interfere with comprehension. On the other hand, reading can signal dissonance to the student who has an opportunity to resolve the dissonance and revise misconceptions. Reading explanations can create opportunities to finish unfinished learning.

5 Mathematics is considered as multisemiotic constructions; that is, discourse formed through choices from the functional sign systems of language, mathematical symbolism, and visual display *(O’Halloran, 2005)*.

Prior knowledge can support or interfere with comprehension. The best outcome in such cases would be to revise prior knowledge. Many so-called "stubborn misconceptions" that plague mathematics comprehension are, in fact, correctly learned bits of mathematics that are being misapplied where they do not apply. A fifth-grade student who tries to understand an explanation for adding and subtracting fractions by using prior knowledge of borrowing and carrying will seem to have a "gap" in knowledge of adding fractions, when it might simply be the misapplication of correct prior knowledge about whole numbers to a situation where it does not apply.

Proofs and Argument

The most common writing challenge in school mathematics is prompted by the words "explain" or "justify." Such prompts, whether from the textbook, on a test, or from the teacher, ask the student to write an informal but well-reasoned argument within the world of mathematics. That means something different from a good argument in other fields and contexts. Ultimately, by the secondary grades, students will be asked to formalize their arguments into proofs. However, in the upper elementary grades, such formalization and abstraction are not appropriate. The following are some desired qualities of proofs and arguments in mathematics:

A well-reasoned argument within the world of mathematics means something different from a good argument in other fields and contexts.

- Less is more. Concise statements bring the logic into the foreground.
- Examples by themselves are not valid evidence, but they can be used to illustrate how something works.
- Counterexamples are a valid way to show a statement is not true. For a mathematical statement to be considered true, **all possible** instances of the statement must be true, **not *almost* all**.
- A useful form of argument is to assume a claim is true and show that the assumption leads to a contradiction.
- The logic that strings claims together is the focus of the writing.
- The justification for each statement can come from a limited set of sources:
  - definitions, properties of operations, equality, inequalities and exponents, rules, already justified propositions (claims), and counterexamples

- "Equals" (=) is the verb in most mathematical sentences (claims). Other examples are "is less than" (<) and "is greater than" (>).
- Definitions play a special role in mathematical reasoning. They limit meaning to precise specifications. It is easier to rely on unambiguous definitions in reasoning: their meaning is restricted in order to convey precision with the language.
- Explicit specification of the domain of numbers being discussed, such as: natural numbers, rational numbers, real numbers, exclusion of 0. This can involve the use of quantifiers like "all," or "there exists a number …," or "for all even numbers …"

Younger students will use concrete examples and objects to reason much more than older students. They explain their insights into how the mathematics works by using illustrative examples. The challenge for them is carrying the argument beyond the examples and explaining why a statement is always true, or true under certain conditions, or never true.

In other words, students have to generalize validly from examples. What students do regularly in talking readies them to do it in writing. If they rarely argue in talk, then they will have a very difficult time writing arguments. They will not know the genre "mathematical justification." They will not know what is wanted when the teacher or the test asks them to justify.

Procedures

We all remember the step-by-step example problem worked out as a model. Our homework was then to follow the example with a set of similar problems. It is a little like learning a new dance each day, rehearsing each night, and then performing. Correspondingly, to this metaphor, students are asked to perform at the end of chapter test several weeks later. An issue here is that teachers will often explain the steps in a procedure, but they may not discuss the conceptual foundations, or even the immediate justification, for the steps. For many, that is what mathematics is. But mathematics is not merely a chain of procedures. A good mathematics student should be able to solve unrehearsed problems that draw on conceptual knowledge and strategic proficiencies as well as procedural competence. Indeed, most uses of mathematics that matter—from state tests to real-life applications—involve unrehearsed problems. If the reader approaches procedural text as a tool for getting the homework done and, later, for rehearsing

If the reader approaches procedural text as a tool for getting the homework done and rehearsing, then that reader is unlikely to comprehend the aspects of the text that would contribute to proficiencies with unrehearsed problems.

for a chapter test, then that reader is unlikely to comprehend the aspects of the text that would contribute to proficiencies with unrehearsed problems. In other words, the mental model of the procedure (if there is one) does not connect to other knowledge in usable ways.

Often, readers of a complicated procedure will say, "I'm lost." They sense that they have lost track of where they have been and where they are going. Feeling lost is a metacognitive signal that comprehension has broken down. Often, the procedure being read refers to sub-procedures "learned" earlier. Here is a typical example from a widely used textbook for adding fractions:

1. Find the least common denominator.
2. Write equivalent fractions using the LCD.
3. Add. Simplify the sum, if possible.

These examples are often considered to be "easy procedure." But these procedures are "easy" only if the reader:

a. has prior knowledge of what a "least common denominator" (LCD) is, which was probably covered 52 pages earlier in the textbook.
b. knows how to find one example, which was taught 50 pages earlier in the textbook.
c. knows what equivalent fractions are, which was most likely taught 44 pages ago in the textbook.
d. remembers how to write them, as well as how to add fractions with common denominators, a topic that is probably buried 10 pages earlier in the textbook.
e. knows what is meant by "simplify" and how to do it.

That is an intricate web of procedural detail scattered over 50 pages of textbook, interrupted by many other topics, not to mention that LCD is unnecessary and more complicated than the main idea of common units (i.e., denominators).

Five hundred pages of procedures per year in fourth and fifth grades are probably too many for anyone. Unless the student is consolidating the mental models, in some way building connections, the text will eventually become dense with obscure references. Students need to learn how to read for conceptual coherence across many pages of text, not just procedural rehearsal.

The Role of Definitions in the Math Genres

Definitions play a special role in mathematics. Definitions are used to justify steps in an argument (they justify claims that lead to other claims). They are more precise than in other fields in a way that can challenge a fourth- or fifth-grade reader. Some people make the distinction that definitions in common language (i.e., dictionary definitions) are empirical—they describe how a word is used; whereas in mathematics, definitions are stipulative—they prescribe how a word is to be used. Understanding the difference is itself a major piece of learning that, especially absent explicit instruction, evades many students, even those who do well according to conventional measures of mathematical achievement.

Mathematical definitions eliminate ambiguity by using a convention that states: the definition means only exactly what it says; it does not mean anything else. Mathematical definitions mean much less than we, as readers, are used to. In fourth and fifth grades, the definitions are often couched in less formal language, but their abstraction and sparseness of meaning are still a challenge. For example, a widely used fifth-grade book has this definition:

> ***Identity Property:*** When 0 is added to any number, the sum is that number.

Of course, the definition is accompanied by examples that make its meaning concrete. Yet students will need to understand the definition as a justification they can use to support their own reasoning and writing. This is a different aspect of understanding from seeing examples.

Mathematical explanations typically use diagrams and specialized notations and symbols. The diagrams often depict a concrete situation being used to make the mathematics less abstract. From the concrete grasp of the idea, the explanation must show how the idea applies to other situations, and where it does not apply. In other words, a particular concrete concept is generalized to a more abstract concept. Good text will use varied examples and analogies to assist the generalization. Often, the more general form of a concept can be most elegantly expressed symbolically. A reader who cannot make good use of the symbolic expressions faces added difficulty in building a mental model of the concept. Engaging with text regularly should lessen this difficulty.

Part II: Supporting Student Engagement with Mathematical Text

When providing complex texts for students, teachers need to consider the overarching concern of accessibility. Very little learning can occur if students cannot make meaning of texts. Since mathematical texts are dense, what scaffolds will students need? Students might be unfamiliar with the distinctive features, the layout, and the conventions of the mathematical texts. Students will need focused language development support that is specific to the discipline. We put forward some approaches that can support students "unpack" mathematical text. Yet we note a word of caution: it is equally detrimental to students when too much scaffolding is offered before students have a sufficient time to get their bearings, formulate their own questions, and do the challenging work of understanding authentic mathematical texts.

We propose an integrated approach of strategies, routines, scaffolds, analysis of language demands, and classroom culture considerations to develop the necessary conditions for students to benefit from engaging with complex texts in mathematics. We also provide some recommendations for math and language teachers to collaborate and tap into each other's expertise. These recommendations can serve as cognitive and linguistic scaffolds to help students negotiate meaning from complex texts.

1. Making Learning Intentions Explicit

Establish a clear purpose for reading, and help readers use information for that purpose. All readers are more efficient and focused when they have a clear purpose for what they want to know or understand from any text. Without a clear purpose, readers will have difficulty distinguishing a driving idea from an example or detail. In mathematics, students' experience with texts is usually quite limited, principally to word problems and computational exercises. If they do read a textbook, it is usually to look for the procedures to solve a problem, not to deepen their conceptual development.

The same mathematical text can be used for different purposes: first, to gain a general understanding of the topic; and then later, to show how different concepts build and relate to each other.

For example:

> **A teacher could begin with a "quick" exercise of adding fractions with unlike denominators. Show each step, following well-established procedures. Hopefully, most will agree with the answer. Now ask: "Why do we need to find common denominators?" Students will explain that it is a rule, or that the denominators have to be same in order to add, or some version of rule-based, circuitous reasoning. Keep pressing for "why?" This leads to a purpose of one of the mathematical texts included herein, *Addition and Length Measurement* by Roger Howe: Why do we need to use common denominators to add fractions? How can we use numbers lines to understand this process? Once the driving question is set, students can read with greater intent, always returning to purpose.**

2. Comparing to Other Disciplines

When introducing mathematical texts, consider creating a display that has various authentic artifacts from other disciplines. For art, you could have a painting; for literature, you could have a poem; for history, you could have a historical document. But leave an empty space for mathematics, and ask students to consider what they might expect to see. Allow them to discuss in groups. They may suggest graphs, equations, formulas, fractions, etc. All these are artifacts in math, but there are also specific texts that mathematicians use to communicate ideas. Just as these other fields of study express their ideas in particular ways, so do mathematicians. From time to time, the class will study these texts together and make sense of them.

3. Relationship among Math Texts

For a selected instructional unit, create a collection of materials that includes the math text you will be working with, but also related lessons from the curriculum, a set of practice problems, an extended project, and a test or assessment that has related content. Ask students to organize these according to purpose. Students will not have much experience in analyzing math texts for the overall purpose, so you can provide some general examples, like showing a recipe whose purpose is to help someone cook something, or showing a map whose purpose is to help someone get from one place to another. Some texts in the collection will have purposes that are more obvious than others. For example, they may say that the purpose of the practice problems is to see if students can get the right answer. This is brainstorming, so allow students to share their ideas. Students may not be clear about the purpose of the math texts. At the end of the activity, tell students that the purpose of these mathematical texts is to understand why and how mathematics works. The purpose is not how to produce an answer to a math exercise, but to understand the logic behind the procedures. You can create a display that has the mathematical text that underlines ideas across all the other texts in the collection.

4. Building Student's Agency

GUIDING QUESTIONS

Teachers can help students learn how to unpack texts by modeling their own thinking and self-monitoring process. We have listed some key questions that teachers can model so that students develop their metacognitive and metalinguistic awareness, especially for how mathematical texts are organized. Of course, teachers should not bombard students with the list of questions below, but rather scan and use them selectively as part of their modeling. This list could be used as a handout to help students see the full range of aspects of texts that skilled readers notice. Below are some questions to promote students' self-regulation while engaging with mathematical text:

- What am I understanding? (Not "Do I understand?") Where am I getting confused? What do I have?
- In places where the author says, "We will assume you know this …," is this true for me?
- Is there something for me to try out? If so, can I do it on my own? Can I get feedback from others?
- What are specific examples and where are the generalizations (*always, for every,* and other quantifiers)?
- Examining figures and illustrations: Do I understand all parts and labels?

- What are the key terms and how are they related to each other?
- What is the author's main point? What confusion is the author trying to clear up?
- Is there an analogy given? Can I visualize it?
- Which terms and definitions seem important? What is the relationship among the key terms that are discussed?
- Can I recreate the math processes and describe them in my own words?
- How do the ideas in one essay connect and build on previous essays?

In working through these questions with students, teachers should model how skilled readers consistently return to the textual evidence. Skilled readers are resourceful. In addition, teachers can model by consulting other reference materials, including dictionaries and glossaries, and conferring with others for ideas.

In addition, student agency could be supported by acknowledging that, in some ways, engaging with math text can feel like problem solving: *What does this mean? What is confusing? What connection does this have with what I already know?* Similarly, the questions listed above can support students to self-monitor their understanding while developing their agency. Providing students with questions that direct their attention to significant features of the mathematical concepts could also be helpful in seeding small-group discussions.

NOTES, MARKINGS, AND ANNOTATIONS

Furthermore, notes, markings, and annotations are strategies that can support student self-monitoring their understanding when engaging with mathematical text. There are many annotation approaches teachers can share with students to track their thinking. Some teachers share highlighting techniques, while others prefer two-column note-taking. Other teachers have students use color-coded sticky notes to affix to sections of the text. A benefit of using sticky notes is that they can be replaced and re-arranged as questions are answered and new questions emerge. The resources below provide useful ideas for students to capture their own thinking:

- *Routines for Reasoning* by Grace Kelemanik, Amy Lucenta, & Susan Janssen Creighton. (2016). Portsmouth, NH: Heinemann.
- *Do I Really Have to Teach Reading? Content Comprehension, Grades 6–12* by Cris Tovani. (2004). Portsmouth, NH: Stenhouse Publishers.

NOTICING AND WONDERING

Distribute copies of the first text you will later explore in depth. For now, ask students to look at the text and describe what they see in brainstorming fashion. They can comment on any feature they notice. Then, ask them how this text is the same and different from their usual math textbook. Collect their ideas to see what they notice. From this list, you can start to assess which features they are noticing, and which ones are still opaque. The point of this initial exercise is not to dig too deep into the content itself, as this is more of an orientation experience. In addition to what students notice, you might also ask if they are having any particular reaction to the text. If students respond with words like "confusing" or "strange," you can add your own reaction as well to show students that it is okay to have those reactions at first. It is a form of wondering.

With sufficient planning time, teachers can explore and decide on the specific strategies they think will work best with students. But we emphasize that strategies themselves are only options for thinking. They are only helpful if students use them in ways that help them understand and unpack complex texts. The students need to decide which strategy or set of strategies is most useful to them. Many times, strategy instruction becomes too prescribed, which students naturally find too restrictive, and thus confirming only a passive role for them. Skilled readers make choices and become more expert the more they practice. Skilled readers often reread and return to texts to deepen their understanding. These are the habits and dispositions that we need to cultivate in students.

The students need to decide which strategy or set of strategies is most useful to them.

PREDICTING: WHERE IS THIS HEADED?

Students, either independently or in collaboration with peers, read the introduction of the mathematical text. Students identify and discuss clues given in the first section that outline which ideas will be developed and why they are important.

SUMMARIZING: WHAT IS THE BOTTOM LINE?

As a product of analyzing a text or part of a text, students create a short summary statement. This can be done in groups as a shared product, which has the benefit of peer interaction, co-construction, speaking, and listening. Groups can then compare their summary statements with others. As time allows, students can refine their statements based on peer and teacher feedback.

CONNECTING: HOW ARE IDEAS CONNECTED?

Besides summarizing the author's ideas from a single text, students need practice with understanding how the ideas within the mathematical texts are connected across essays. What are the threads of reasoning and representations that were developed across texts? Students could discuss this in small groups, citing textual evidence to support their claims. Ideally, what we would want students to notice is the underlying unity of mathematics and mathematical structures. For example, even though we appear to use different procedures for working with fractions, whole numbers and integers, in fact, when we consider them as lengths on a number line, they share all the same properties.

REFLECTING: WHAT HELPED ME LEARN?

Negotiating meaning from complex texts is a major accomplishment, requiring both the effort and persistence of any learner. Although we have a rich literature of metacognitive practices, we see little evidence of their actual classroom use in mathematics. In classrooms with heavy emphasis on skill development, perhaps this is a logical consequence. But in classrooms that are more balanced, where conceptual understanding is more salient, metacognitive practices can help students increase self-awareness and self-regulation. Students can reflect in writing on a number of questions, such as: How did I handle my confusion? What helped me most? How was I helpful to others? What can I improve on next time?

5. Analyzing the Demands of Academic Language in Mathematics

Academic language[6] can present a challenge for all students, particularly if they have limited experience and instruction with it. In our experience, more attention should be given to the explicit teaching of academic language in the math classroom.

Teachers who specialize in mathematics are best positioned to support students "unpacking" the academic language of their discipline. Rather than focusing only on academic vocabulary, an approach that emphasizes "functional language analysis" can be much more effective in teaching students how mathematicians use language *(Fang & Schleppegrell, 2010, p. 591)*.

Although success in mathematics depends greatly on content knowledge, academic language and literacy skills present major obstacles to the learning of content knowledge from authentic texts *(Fang, 2017, p. 333)*. Mathematical discourse is technical, dense, and multisemiotic. It draws from natural language, symbolic language, and visual display in a synergistic way *(Fang & Schleppegrell, 2010, p. 591)*.

6 Academic language is the specialized language, both oral and written, of academic settings that facilitate communication and thinking about disciplinary content *(Nagy & Townsend, 2012)*.

We recommend that teachers analyze the linguistic demands of mathematical text prior to engaging students with it. While analyzing the language demands of mathematical text can be an extensive topic, students can benefit from explicit attention to at least three features of academic language: (a) dense information, (b) connecting ideas logically, and (c) tracking ideas. Below are some examples (see Table 2).

a. Dense information:

While density can be evident in academic text through several language features, we will narrow our attention to a few selected structures of academic vocabulary and grammatical patterns that are prevalent in mathematical text. We suggest that a good starting point is for students to acquire skills with "unpacking" morphologically **complex words** and complex sentences. One example of metamorphically complex words in math are words that consist of a root and one or more affixes (i.e., prefix and suffix). Technical words in math (see Table 2) are often the result of prefixes and suffixes (e.g., polynomial, diameter, and apothem). In addition, density often results in mathematics from abstractions conveyed by nominalizations[7] (see example in Table 2).

Furthermore, the density of mathematical text is often the result of **complex sentences**. In particular, mathematical texts have the tendency to exploit long noun phrases (Table 2). Students will need support recognizing and becoming familiar with language patterns that exploit long noun phrases to facilitate the construction of definitions, theorems, propositions, lemmas, corollaries, and proofs (see Table 2).

b. Connecting ideas logically:

Students' attention should be directed to implications of math-relevant connectives and discourse markers. That is, learning to recognize markers that signal logical semantic relations that connect ideas (see Table 2).

c. Tracking mathematical ideas:

Students need to develop facility in recognizing words and phrases appearing in a text that refer to a prior mathematical idea in a referential chain (see Table 2).

7 The process of turning a verb or adjective into a noun, typically, but not always, by adding a suffix *(Nagy W., Townsend D., 2012)*.

Table 2: Features of Academic Language

Features	Examples in General	Examples in Mathematics
Technical Words Academic language tends to contain longer words, and for the most part, the length is due to prefixes and suffixes	darkness, unlucky, subway	numerator, denominator, polygon, quadratic, analysis, extrapolate, correlation
Nominalization Nouns that are created from clauses, phrases, adjectives, and verbs	The **marker** for the road is missing.	Experience with the operations of **addition and multiplication** leads to the observation of certain regularities in their behavior.
Long noun phrases A group of words that function like a noun	**The small brown dog** has no leash.	**A function of the form** $y = ax^2 + 2bx + c$, where $a \neq 0$, is a quadratic function.
Connecting ideas logically Skills in understanding school-relevant connectives and discourse markers	because, but, and	therefore, imply, hence, if-then, if and only if, caused, consequently, results, equal, equivalent (=), greater than (>), less than (<)
Tracking ideas Anaphors (i.e., words or phrases appearing in a text that refer to a prior participant or idea) and referential chain	She refers to Mary.	This procedure (e.g., $\sum_{i=1}^{n} i$), The statement (e.g., $a = b + 1$) This relationship (e.g., $x \in A$)

Messaging to Students the Expectation for Engaging with Mathematical Texts

To make these mathematical texts a regular part of math instruction, it is of paramount importance to attend to the culture of the math classroom. Mathematics, as it is experienced in the classroom, is typically a stressful subject for students, and an anxious one for many others. Even for many students who do get good grades, it is difficult to know what these students truly understand from traditional assessments. Introducing complex mathematical texts will certainly add to this initial feeling of confusion, as students will have had no experience with authentic mathematical texts. So, expect confusion and even a certain level of resistance. Being confused is uncomfortable for many apprentice learners, yet we know that confusion is a necessary state for new learning to occur.

Below we have listed key messages that are appropriate for any mathematics classroom, but are especially salient when working with complex mathematical texts. Over time, these messages can become norms of the classroom. They can also be reworded in ways that are more understandable to students at different grade levels.

Key Messages to Give Students:

- The purpose of mathematical texts is to help you understand how and why everything fits together in math! It is not just about formulas and procedures you need to follow.
- The part of texts that is confusing is exactly where new learning will occur! Asking questions is what mathematical thinkers do. We will work together to keep track of your questions and find satisfying answers.
- Slow and careful reading will help you understand the ideas and connections! This is not a race.
- Sharing our ideas, even if we are not 100 percent sure, is useful because we get to test out and refine ideas, or completely change our minds. We can also get helpful ideas from listening to others who may see things a different way. Mathematics is created and developed together.

Many teachers with the best intentions will post such a list. However, more important than posting the list is clearly and consistently maintaining these norms throughout the year. One way to reinforce norms is to identify times in the classroom where teachers see the norms at work. For example, after a class in which students struggled to understand the content, teachers can recognize their efforts and let students know that this is challenging work, and that they were pleased to see students wrestle with these texts. Alternatively, if teachers see behavior that is contrary to these messages, they can also let the class know. For example, "I noticed that some of you were trying to speed-read. That's a good strategy for getting a general sense, but remember that we need slow and careful thinking. So, the next time, let's not be so rushed." Of course, teachers will also want to be sure that they are giving students enough time in class for careful thinking.

Over time, these ways of working will become more familiar. Students will build habits to decipher the math texts and to understand that they may need to re-read passages several times.

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Conclusion

The mathematical texts we discussed will provide all students with grade-appropriate opportunities to negotiate meaning around rigorous mathematical ideas. As this represents an innovative departure from the readings that students typically do in math class, we expect that math teachers at all grade levels will need more support in their professional learning communities about how to best engage students with math texts in their curriculum.

We encourage math teachers to use their wealth of knowledge and professional experiences to refine and adapt many of the recommendations we have outlined in this document, as well as to identify additional math texts that support students in developing content knowledge. We encourage collaboration of math teachers to work together to identify and engage students with mathematical text in their curriculum as well as other sources. We encourage math teachers to implement their units using one of the math texts, to collect samples of students' work and audio samples along with teacher reflections, and to share their data with other teachers. We encourage math teachers to film themselves and to use their videos to create case studies to be shared with other math teachers.

Part III: A Collection of Authentic Expository Mathematical Text

This section offers a collection of expository mathematical texts about important mathematical ideas. The collection of texts is written by a mathematician on a variety of topics for students in several grade levels. Each text is mapped to the New York Next Generation Math Standards.

Because the texts address foundational topics in mathematics, they offer teachers much flexibility and freedom in using them. Some potential uses of this collection are to:

- Activate prior knowledge.
- Build student agency.
- Deepen mathematical knowledge.
- Develop academic language in tandem with content.
- Help students make sense of the progression of mathematical ideas through the grades, ie: connecting properties of operations to length measurement.
- Model to students the use of mathematical language in written discourse.
- Promote oral discourse.
- Supplement units of study.
- Support students with missed learning opportunities in mathematics.

Substantial effort was put into calibrating the reading level and the content to create texts that are accessible for students in upper elementary, middle, and high school levels, while still preserving the distinctive features of mathematical exposition. Future efforts will be directed to increase this collection to include more topics and grades. Feedback, both specific and general, regarding points of difficulty or value will be welcomed, and can be sent to **STEM@schools.nyc.gov**.

Table: Mapping of Expository Texts to NYS Next Generation Standards

Title	NYS Standards (Click here to see NY NGLS.)	Text
1. Putting Fractions on the Number Line—Multiples of a Unit Fraction	NY-3.NF.1 → NY-4.NF.3 → (NY-5.NF.1 and NY-5.NF.2) NY-4.NF.6 → NY-5.NBT.3a NY-3.NF.3d → NY-4.NF.2 NY-3.NF.3 → NY-4.NF.1 → NY-5.NF.1	The number line is a terrific tool for helping us understand arithmetic in terms of geometry. This can help you get a much more unified picture of what arithmetic is about than working only in symbols, and can help arithmetic make more sense ... Read Full Essay
2. Fractions on the Number Line—Dealing with Different Denominators	NY-3.NF.1 → NY-4.NF.3 → (NY-5.NF.1 and NY-5.NF.2) NY-4.NF.6 → NY-5.NBT.3a NY-3.NF.3d → NY-4.NF.2 NY-3.NF.3 → NY-4.NF.1 → NY-5.NF.1	In using fractions, one of the difficult things is the idea of equivalent fractions. Our base ten system gives a unique name for every whole number, but our symbolic method for dealing with fractions is very redundant … Read Full Essay
3. Addition and Length Measurement	NY-3.NF.1 → NY-4.NF.3 → (NY-5.NF.1 and NY-5.NF.2)	Addition is a basic operation of arithmetic, and for whole numbers, is not hard for most students to understand, but addition of fractions may seem confusing … Read Full Essay

Title	NYS Standards (Click here to see NY NGLS.)	Text
4. Subtraction and Length Measurement	(NY-6.NS.5 and NY-6.NS.6 and NY-6.NS.7) → NY-7. NS.1NY-6.NS.7)	Subtraction is closely related to addition. Sometimes it is called "the opposite of addition." It is more correct to say that subtracting a given number is the opposite of adding that number … Read Full Essay
5. Addition on the Number Line—Positive Numbers	(NY-2.NBT.5 and NY-2.NBT.7) → NY-3.NBT.2 → NY-4.NBT.4 NY-6.NS.5 → NY 7.NS.1	The basic idea is that a number labeling a point tells you how far the point is from the origin (that is, 0), as a multiple of the unit length, which is the distance between 0 and 1. Read Full Essay
6. Subtraction on the Number Line	(NY-2.NBT.5 and NY-2.NBT.7) → NY-3.NBT.2 → NY-4.NBT.4 NY-6.NS.5 → NY 7.NS.1	For subtraction, we don't put the bars end to end. We put them next to each other, with one end of each matching an end of the other, and find the length of the overhang. In fact, our bars are already next to each other, with their left ends (at 0) matching … Read Full Text.

Title	NYS Standards (Click here to see NY NGLS.)	Text
7. Putting Numbers on the Number Line—Signed (Positive and Negative) Numbers	NY-5.G.1 → NY-6.NS.6 → NY-7.NS.1	Here we will discuss putting both positive and negative numbers on the number line. You should already understand how to place positive numbers. The main emphasis here will be on extending this process to negative numbers ... Read Full Text.
8. Addition on the Number Line—Signed Numbers	NY-6.NS.1 → NY-7.NS.2 → NY-8.NS.1	The key thing to understand in adding signed numbers on the number line is the role of orientation, or direction. When we deal with the whole number line, it is divided into two half lines by 0, the origin… Read Full Text.
9. The Negative and the Number Line	NY-5.NBT.3b → NY-6.NS.7 → NY-7.NS.1	It is important to keep in mind that that the word "negative" gets used in two different ways, although they are related. When we say, "the negative" or "the negative of a number," we mean … Read Full Text.

Title	NYS Standards (Click here to see NY NGLS.)	Text
10. Unifying Addition and Subtraction	(NY-6.NS.5 and NY-6.NS.6 and NY-6.NS.7) → NY-7.NS.1	Addition and subtraction are linked from the beginning of the study of arithmetic. But some students never learn just how closely they are connected. In some sense, subtraction is (almost) included in addition … Read Full Text.
11. Basics of the Place Value System	NY-4.NBT.1 → NY-5.NBT.1	The place value notation that we use to express whole numbers is an extremely clever and efficient method. It took thousands of years and several stages of improvement before it reached the sophisticated form that we use today ... Read Full Text.
12. The Size of Base Ten Numbers	NY-4.NBT.5 → NY-5.NBT.5 → NY-6.NS.3	One of the nice features of our base ten place value system for writing numbers is that it makes it easy to understand roughly how big numbers are, and it also makes comparing numbers very easy … Read Full Text.

Title	NYS Standards (Click here to see NY NGLS.)	Text
13. Comparison of Base Ten Numbers	NY-4.NBT.2b NY-4.NF.7 → NY-5.NBT.3b → NY-6.NS.7	If a first number is less than a second number, and if the second number is less than a third number, then the first number is less than the third number … Read Full Text.
14. The Size of Base Ten Numbers and the Number Line	NY-4.NBT.2b NY-4.NF.7 → NY-5.NBT.3b → NY-6.NS.7	As you move to the right in the places, the place value parts get smaller and smaller. In fact, they get smaller very fast, because each base ten unit is ten times as large as the next smaller one. This feature has the advantage that it allows us to express … Read Full Text.
15. Putting Decimals on the Number Line	NY-4.NF.7 → NY-5.NBT.3b	... the contribution of the remaining place value parts makes comparatively little difference in the size of the number. It is also worthwhile to study this phenomenon from a numerical viewpoint ... Read Full Text.

Title	NYS Standards (Click here to see NY NGLS.)	Text
16. Approximating Numbers by Rounding	NY-2.OA.3 → NY-3.OA.9 → NY-4.OA.5	A more refined statement of the change created by rounding is that the difference between rounding up and rounding down at a given place is exactly one base ten unit corresponding to that place. For our examples above, this amounts to the inequalities … Read Full Text.
17. The Number Line and the Size of Fractions	NY-2.MD.6 → NY-3.NF. → [NY-5.NF + NY-5.MD] NY-4.MD.4 → NY-5.MD.2 → [NY-6.SP.2 and NY-6.SP.2] NY-5.NF.3 → NY-6.RP.2 → NY-7.RP.1 [NY-6.RP.2 and NY-6.RP.3b] → [NY-7.RP.1 and NY-7.EE.3] NY-6.NS.1 → NY-7.NS.2 → NY-8.NS.1	Fractions cause lots of problems for lots of students. One obstacle for many students is figuring out how large a fraction is. This is related to the task of comparing fractions—deciding which one of two fractions is the larger one. Thinking about putting fractions on the number line can help with these problems … Read Full Text.
18. What Are Functions?	NY-7.RP.2b →NY-8.F.2 → A1-F.IF.9 NY-8.F.1 → A1-F.IF.1 → A2F.IF.7e	… a typical point in the plane is an ordered pair (x, y) of numbers. The points where $y = f(x)$, that is, the points $(x, f(x))$, give a curve in the plane, with one point on each vertical line whose x coordinate is in the set of inputs … Read Full Text.

1. Putting Fractions on the Number Line—Multiples of a Unit Fraction

The number line is a terrific tool for helping us understand arithmetic in terms of geometry. This can help you get a much more unified picture of what arithmetic is about than working only in symbols, and it can help arithmetic make more sense. The first step in getting this better understanding is knowing how to put fractions on the number line. So, let's start with that.

Positive Numbers

Let's talk about positive numbers first.

You have probably seen the number line. It is often drawn something like this.

The most important thing to realize about the number line is that it is basically an idealized ruler. It is about distance or length. We will be using these two words, "distance" and "length," a lot. So, let's start by being clear about how they are related.

Length refers to the size of a line, or a thin rod, or a thing that is long and thin. For example, the line on the left below is one inch long, and the thin strip next to it is also one inch long. The shorter line next to that is one centimeter long, and the thin strip next to that is two centimeters long.

If we have two points, we can imagine drawing the straight line that connects them. The **distance** between the points means the length of the line between them. So, if we label the endpoints of the lines or bars in the picture below:

Then, the distance between A and B is one inch, and so is the distance between C and D. The distance between E and F is one centimeter, and the distance between G and H is two centimeters. In general, the length of a straight line is the distance between its endpoints, and vice versa.

You know that to measure length, you need to choose a **unit** of length: inch, centimeter, foot, meter, or other. This is part of constructing a number line. Although you have probably seen

pictures of the number line like the one above, it may be that you have never seen the process of constructing one. Below is a short description of how that is done.

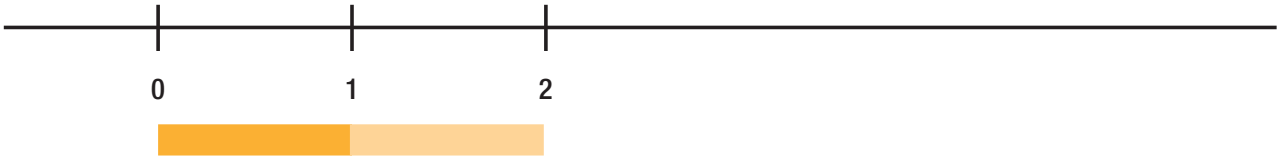
You start with an unmarked straight line:

Next, you choose a point to be the origin, or zero point. Since we are only worrying about positive numbers at the moment, we will put the zero point on the left.

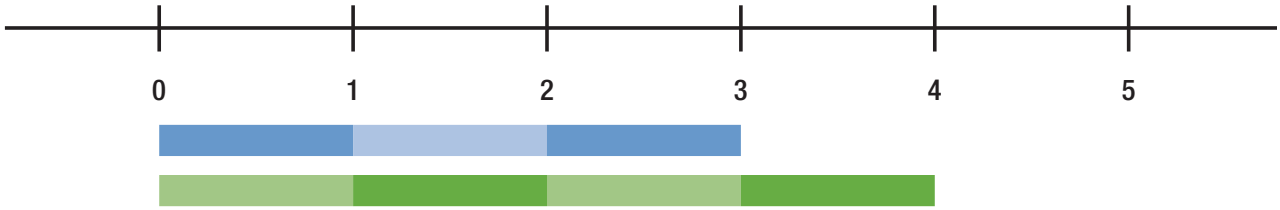
Next, you choose a point to be 1:

This sets the unit length: on a number line, the unit length is the length of the interval between 0 and 1.

From here on, the position of every other number is determined. The number labeling some other point tells you the distance between that point and the origin, as a multiple of the unit distance. For example, the point labeled by 2 is at distance of 2 from the origin. You can see this, because you can fit two intervals of length 1 between 0 and 2. This is shown in the picture below. The intervals are different colors, so that it is easy to tell them apart, but they both have length 1.

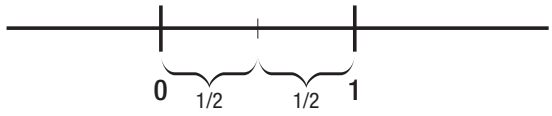


In the same way, you can fit three unit intervals in the space between 0 and 3, and four unit intervals in the space between 0 and 4, and so on.

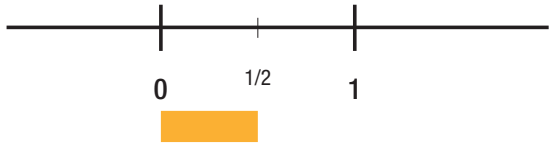


The same relationship with length measurement also tells us how to put fractions on the number line. For example, 1/2. To get an interval of length 1/2, we cut the unit length

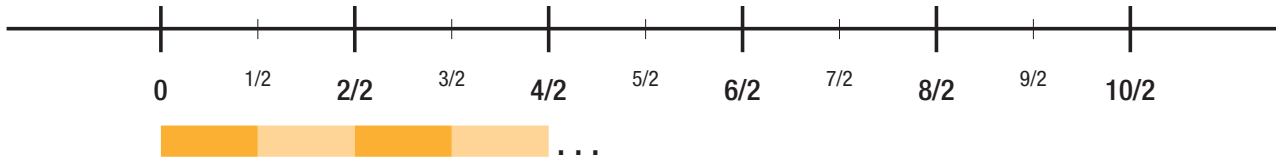
into two equal pieces. Each one has length 1/2:



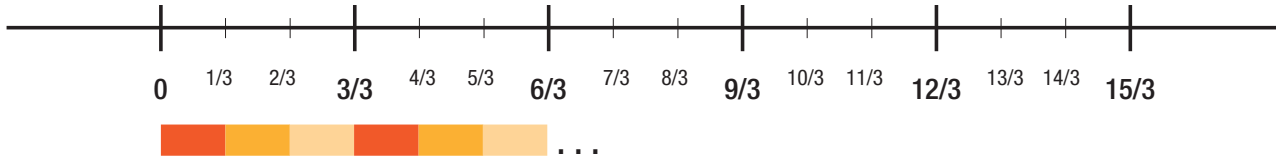
Then, according to the rule relating numbers to distance, 1/2 should label the point at the right end of an interval of length 1/2, with its left end at 0.



Following up, we would put 2/2 at the distance made by two bars of length 1/2 . Of course, this is the same as 1, which you learned in arithmetic: 2/2 = 1. Continuing, we would put 3/2 at distance equal to 3 lengths of 1/2, and 4/2 would go 4 lengths of 1/2, or 2 unit lengths, from 0. Continuing this process, we can put all the whole-number multiples of 1/2 on the number line. They make a regular array, just like the whole numbers that we started with above, but only half as far apart.

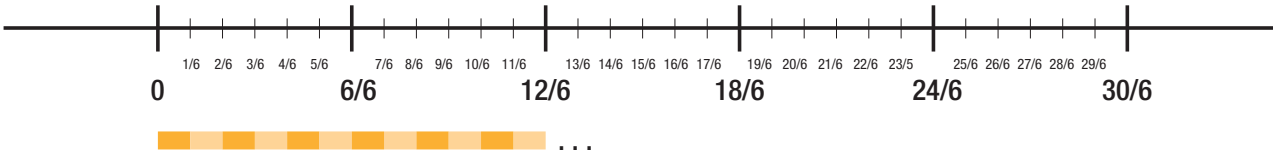


The same idea applies to fractions with larger denominators. To get 1/3 , we divide the unit length into 3 equal pieces. Then we take one of those pieces and put one end (the left end, in our pictures) at 0, and the other end will be at 1/3. To get 2/3, we take two of these lengths and put them together, with one end of the combined length at 0, and then the other end will be at 2/3. Continuing in the same way, we can mark off all the multiples of 1/3 :



The picture is similar to the number line with only the whole numbers marked, and the number line with the multiples of 1/2 marked: an evenly spaced set. However, the multiples of 1/3 are only 1/3 as far apart as the whole numbers, with three intervals of length 1/3 in every unit interval. It would be similar with 1/4 and its multiples. Try drawing that one for yourself.

The same ideas would give all the whole-number multiples of any unit fraction. If the denominator is d, the multiples of 1/d are an evenly spaced set of points on the number line, and chopping up each unit interval into d equals smaller intervals. Here is the picture when d = 6.



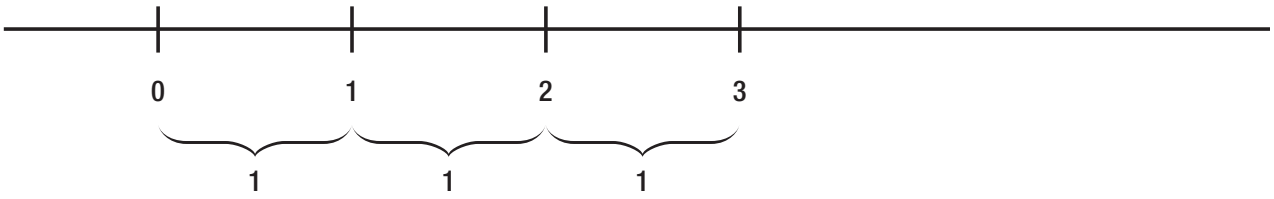
BACK TO TABLE

2. Fractions on the Number Line—Dealing with Different Denominators

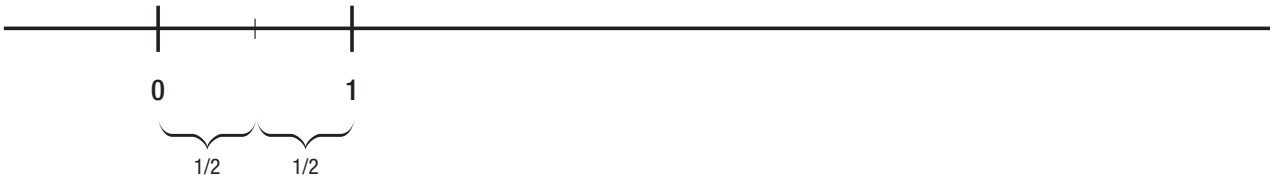
In using fractions, one of the difficult things is the idea of equivalent fractions. Our base ten system gives a unique name for every whole number, but our symbolic method for dealing with fractions is very redundant—there are an infinite number of fraction symbols that name any individual rational number. This can make working with fractions confusing, and certainly makes it harder. But since no one has thought of a scheme that avoids this redundancy, we have to deal with it. Fortunately, the number line helps us see what is going on with equivalent fractions.

To read about this, you should already understand the basics of how we put numbers on the number line, and especially that it is based on length and distance. You determine a unit length by choosing the 0 and 1.

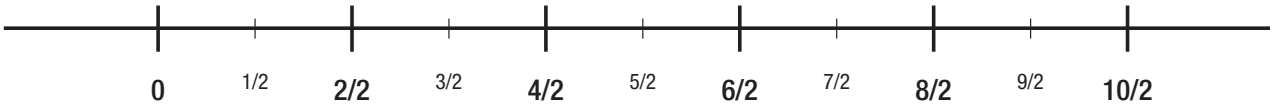
Then every other number tells you the distance of a point from 0, as a multiple of the unit length. For example, 3 labels the point that is three times as far from 0 as 1 is,



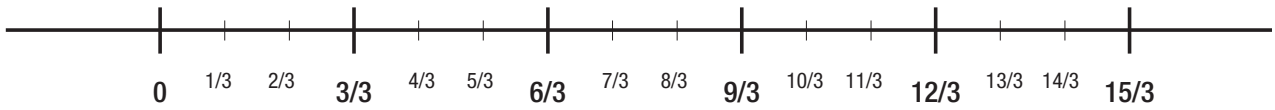
and 1/2 labels the point that is only half as far from 0 as 1 is.



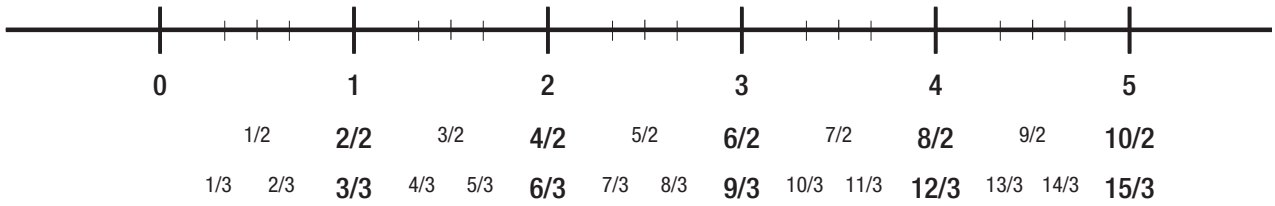
Using these ideas, it is easy to construct the system of all whole number multiples of a given unit fraction. Here is the system of multiples of 1/2:



This looks almost like the whole numbers—evenly spaced points—but with the separation only half as much as for the whole numbers, and two intervals in every unit interval. The system of 1/3s is similar, with evenly spaced points, but now they are only 1/3 as far apart as whole numbers, and three of the intervals separating them fit into a unit interval.

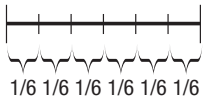


This process will put all fractions on the number line, but it may give the impression that this just creates an infinite number of systems, one for each denominator. And although the system for any fixed denominator is very regular, the systems for different denominators may not fit together so well. Look at this number line that has all the multiples of 1/2 and all the multiples of 1/3.



You can see that the intervals between the points are different lengths. You might try doing the same thing with multiples of 1/3 and 1/4, or 1/4 and 1/5. The spacings can be quite different.

What to do? It turns out, there is a pretty simple answer. The systems for 1/2 and 1/3 both fit inside the system for 1/6! This is because 6 is a multiple of both 2 and 3. Since 6 sixths make up a whole:



we can see that a group of 3 sixths make 1/2. Likewise, a group of 2 sixths make 1/3. In other words,

$$1/2 = 3/6 \quad \text{and} \quad 1/3 = 2/6.$$

We call fractions that look different, but are equal, that is, that represent the same number, **equivalent fractions**.

So both the system of halves and the system of thirds fit inside the system of sixths! This will work with any two fractions. Both will fit inside the system of multiples of a unit fraction whose denominator is divisible by the denominators of the two given fractions. For example, the denominator that is the product of the denominators of the two given fractions will work. This lets us deal with many questions of arithmetic; for example, addition and subtraction of fractions. Exactly how this works is a story for another time.

A valuable thing to keep in mind about the relationships in this discussion is, that the relations between the fractions involved is in some sense opposite to the relation between the denominators. For example, 6 is a whole number multiple of 2, but 1/2 is a whole number multiple of 1/6: $6 = 3\times2$, while $1/2 = 3\times(1/6)$. Similarly, $6 = 2\times3$, but $1/3 = 2\times(1/6)$. And although 6 is larger than 2 or 3, both 1/2 and 1/3 are larger than 1/6.

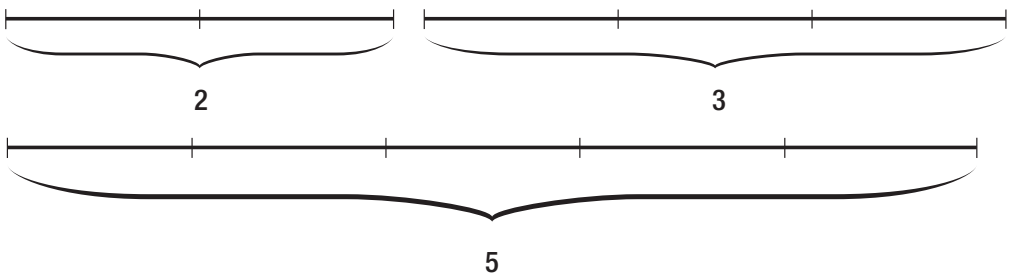
BACK TO TABLE

3. Addition and Length Measurement

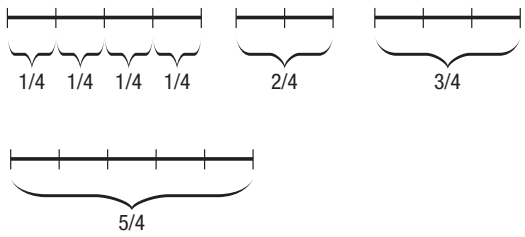
Addition is a basic operation of arithmetic, and for whole numbers, is not hard for most students to understand, but addition of fractions may seem confusing. Certainly, it is complicated at a symbolic level, and involves finding new denominators, and often requires multiplication. The number line can help us understand addition of any two numbers in a uniform way, and also why the denominators need changing. Let's look at how this works.

The number line is about length, so what does addition look like in the context of length measurement? If we think of two lengths as being given by two bars, then the sum corresponds to the bar that results from putting the two given bars together, end to end. To measure length, we need a unit of length. Suppose this is our unit length:

Then, if we put a bar of length 2 next to a bar of length 3, we can create a bar of length 5.

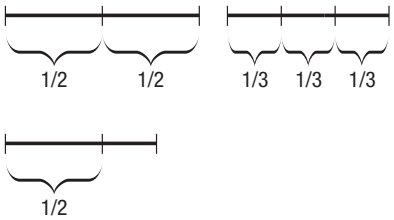


In fact, this process tells us how to add, not just whole numbers, but any numbers, including fractions. For example, suppose we want to add 2/4 + 3/4. Then we take bars of lengths 2/4, and 3/4, and join them end to end. The result is a bar of length 5/4.

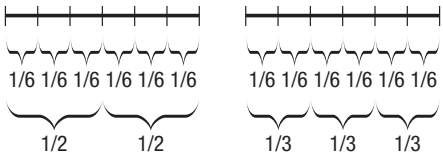


This example illustrates the general fact that if we want to add two fractions with the same denominator, we just have to add the numerators. The denominator stays the same. This part of fraction addition is relatively easy to deal with.

The geometric process of joining bars applies to any two fractions, not just to ones that are both expressed as multiples of a fixed unit fraction. For example, let's look at 1/2 + 1/3 . We can take bars of length 1/2 units and 1/3 units and put them together, to get a bar whose length will be 1/2 + 1/3 units.



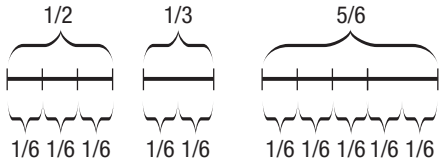
This gives us the answer as a length. But it doesn't tell us how to name the length as a fraction. How can we do that? The difficulty is related to the fact is that the fraction 1/2 and 1/3 each refer to one piece, but the pieces are of different sizes. When we are adding whole numbers in the usual way, all the numbers we add should refer to pieces of the same size, or have the same unit. So we should try to express 1/2 and 1/3 in terms of a common unit. To find a common unit, we use the fact that both 1/2 and 1/3 can be expressed in terms of sixths. Precisely, 1/2 = 3/6 and 1/3 = 2/6.



Now if we put our bars together, we see that their total length is

$$1/2 + 1/3 = 3/6 + 2/6 = 5/6.$$

This is the usual result found by symbolic work.



An important point to understand is that the top and bottom of a fraction, the numerator and the denominator, are telling us very different things. The top is telling us the number of parts, and the bottom is telling us about the size of those parts. But, to repeat: when we add, we want to know that the numbers we are adding are all referring to the same sized piece, also called the whole or the unit. Fortunately, we can always express two fractions as being equal to fractions with a common denominator. This common denominator can always be taken to be the product of the denominators of the two fractions. Sometimes, a smaller number also works.

4. Subtraction and Length Measurement

Subtraction is closely related to addition. Sometimes it is called “the opposite of addition.” It is more correct to say that subtracting a given number is the opposite of adding that number. In other words, if you add a number to another one, then subtract it, you get back to the other one. So (1 + 2) - 2 = 1. You don’t even have to compute the sum of the numbers in parentheses, you can immediately write the answer. So also you can be sure that

$$(7{,}632{,}488 + 5{,}117{,}396) - 5{,}117{,}396 = 7{,}632{,}488,$$

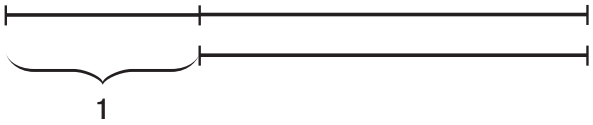
without ever calculating the sum.

You should already know that addition can be interpreted in terms of length: it is the combination of lengths, or placing bars end-to-end to make longer bars. For example, suppose that this is a bar of length 1 (in other words, it is the unit of length):

Then here are bars of length 1 and length 2:

If we put them together, we get a bar of length 3:

To get back to the bar of length 1, we need to get rid of the bar of length 2. But if we just make it disappear, how do we know what has happened? To make the subtraction visible, we put a second bar of length 2 below the bar of length 3:



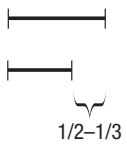
Then the length of the unmatched part of the bar of length 3 is the difference 3 - 2. We call this process, of putting a shorter bar next to a longer bar, with ends matched up, and looking at the overhang, **length comparison**.

To summarize, in length measurement, addition corresponds to the process of combining lengths, and subtraction corresponds to the process of length comparison.

Just as length combination provides a uniform way of understanding addition of any kind of numbers—in particular, fractions as well as whole numbers—length comparison is just as applicable to subtraction of fractions as it is to subtraction of whole numbers.

As an example of this, let's look at 1/2 - 1/3. We can find lengths of 1/2 and 1/3 by subdividing the unit length into 2 or 3 equal parts, and taking one of them.

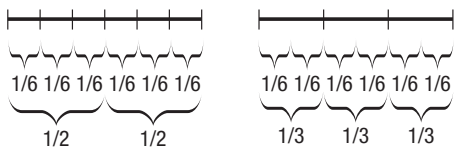
Then we subtract 1/3 from 1/2 by comparing these two lengths.



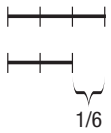
This is the answer, but we would like to also be able to express it in symbolic form. For this, we will assume that you know that both 1/2 and 1/3 can be expressed as multiples of 1/6. Precisely,

$$1/2 = 3\ (1/6) \qquad \text{and} \qquad 1/3 = 2\ (1/6).$$

These relationships are illustrated in these pictures.



Now, if we break up the lengths of 1/2 and 1/3 into sixths, and compare them, we can see that the overhang is exactly 1/6.



This lets us conclude that

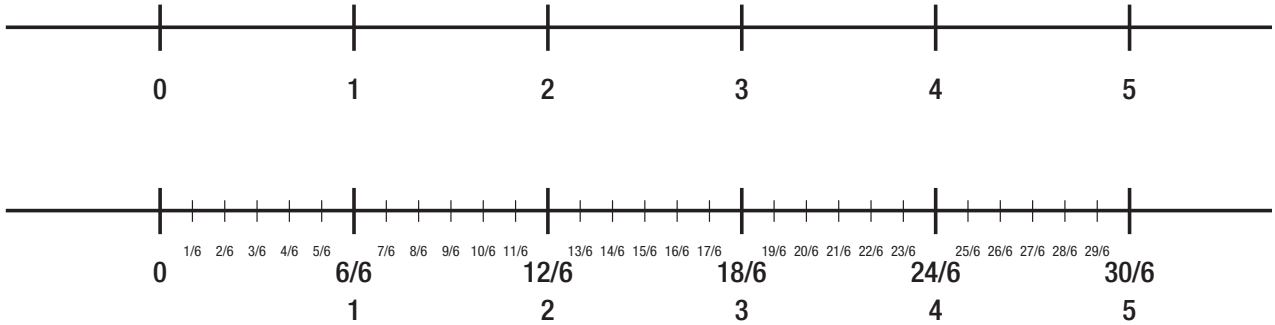
$$1/2 - 1/3 = 3/6 - 2/6 = 1/6.$$

This method will work with any fractions. Try a few examples.

BACK TO TABLE

5. Addition on the Number Line—Positive Numbers

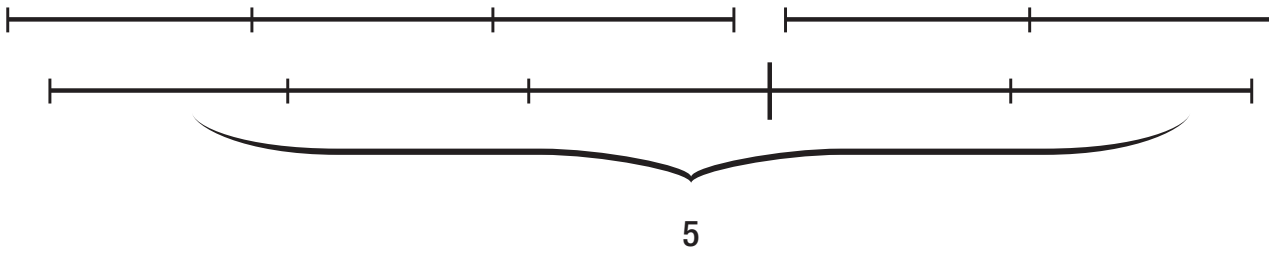
For this discussion, you should know how to put positive numbers on the number line, both whole numbers and fractions. The basic idea is that a number labeling a point tells you how far the point is from the origin (that is, 0) as a multiple of the unit length, which is the distance between 0 and 1. With this rule, the fractions with a fixed denominator make a regularly spaced array of points, like the whole numbers, but closer together. Pictured here is the line with the whole numbers marked, and below it, with all the multiples of 1/6 marked.



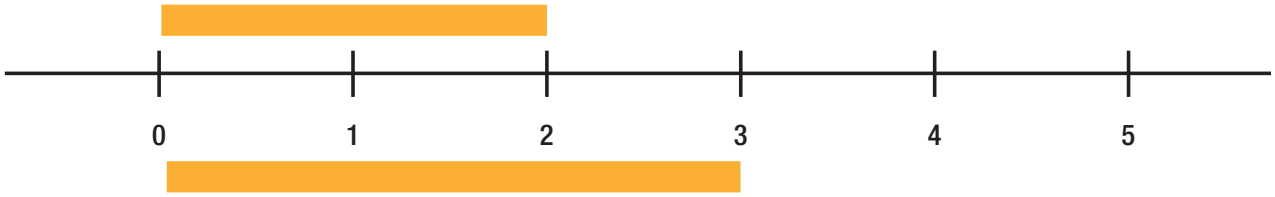
You should also understand how to interpret addition using length measurement: addition amounts to putting lengths together, end to end. The length measurement version of addition has the advantage that it is just the same for whole numbers and for fractions, while the symbolic versions may seem quite different.

Now we want to see how to interpret addition using the number line. This is a little harder work, but it has the advantage that it can be extended to adding signed numbers (meaning negative numbers as well as positive numbers), which confuses many people.

So suppose we want to find 3 + 2. Using length measurement, we would take a line of length 3 and a line of length 2, and put them together to get a line of length 5, showing that 3 + 2 = 5.

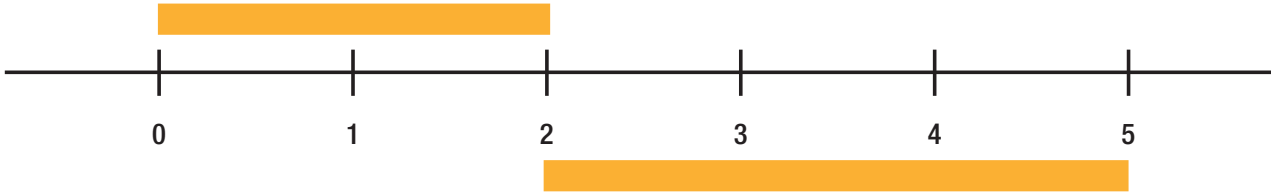


How would we do this using the number line? The point is, the number line gives us a standard example of a line of any length: it is just the interval between 0, the origin, and the number giving the length. So we already have lengths of 3 and 2 on the number line:

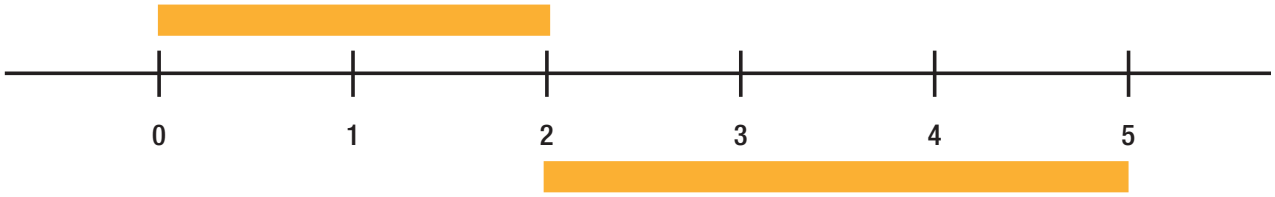


The problem with this situation is that we want to place the two lines end-to-end, but at the moment, they are overlapping each other.

What we have to do is move one of the bars along the line, until its left end matches the right end of the other one. Then the right end of the bar will stretch to the number that is the sum of the two numbers.



We have to choose which one of the bars to move. In this illustration, we have left the line of length 3 fixed, and moved the line of length 2, because we wrote the sum as 3 + 2. But we know that 3 + 2 = 2 + 3, because addition is commutative. So we could have moved the other bar, and the result would be the same.



Before electronic calculators were invented, engineers had a tool called a slide rule, which essentially consisted of two number lines, one fixed and one slidable, so they could do addition by the process just illustrated. Of course, they had to deal with numbers more complicated than 2 and 3, so it was good for them that this process works with any kinds of numbers.

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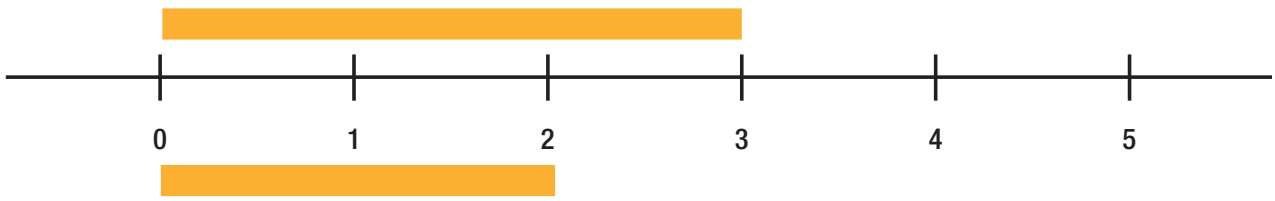
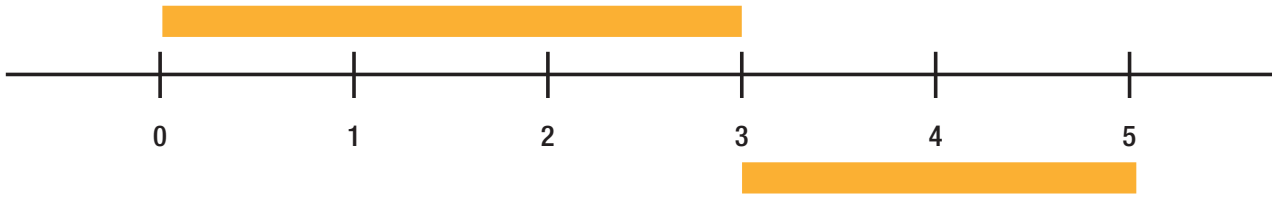
6. Subtraction on the Number Line—Positive Numbers

To read this note, you should understand how to place numbers on the number line, and you should understand how to do subtraction with length measurement. Briefly, a number labeling a point on the positive number line tells you how far that point is from the origin, as a multiple of the unit length (the distance from 0 to 1). Using length measurement, subtraction amounts to comparing the length of lines (or bars). You place the lines of two length side by side, with one end of each aligned with an end of the other. Then the difference is the length of the "overhang"—the part of the longer line that is not covered by the shorter one.

We want to adapt this subtraction process to the number line. In the first paragraph, we described a key feature of the number line—for any length, it provides a standard example of a line of that length, namely, the interval between 0 and the number describing that length. So, if we want to do a subtraction, we should start with these two lengths. Since you already know that the process of subtraction using length measurement is independent of the specific numbers involved, let's work with a very simple example, say 3 - 2. Here is the number line, with lines above and below showing the standard bars of length 3 and length 2.



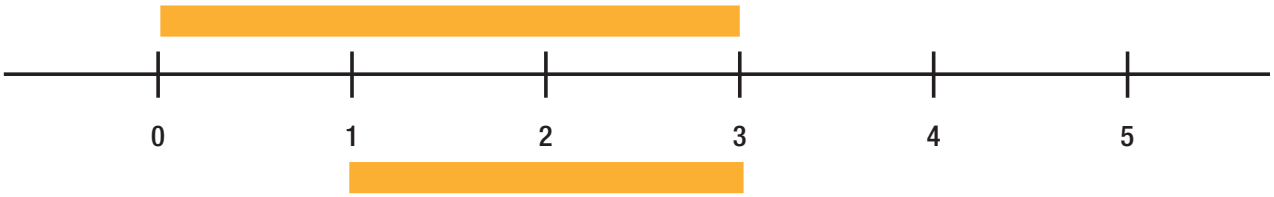
If we want to add 3 and 2, we need to put the bars end to end. We can do this by sliding one of the bars along the line until its left end matches the right end of the other bar. If we do this, then the right endpoint of the line that we moved is at the sum. We just have to read it off.



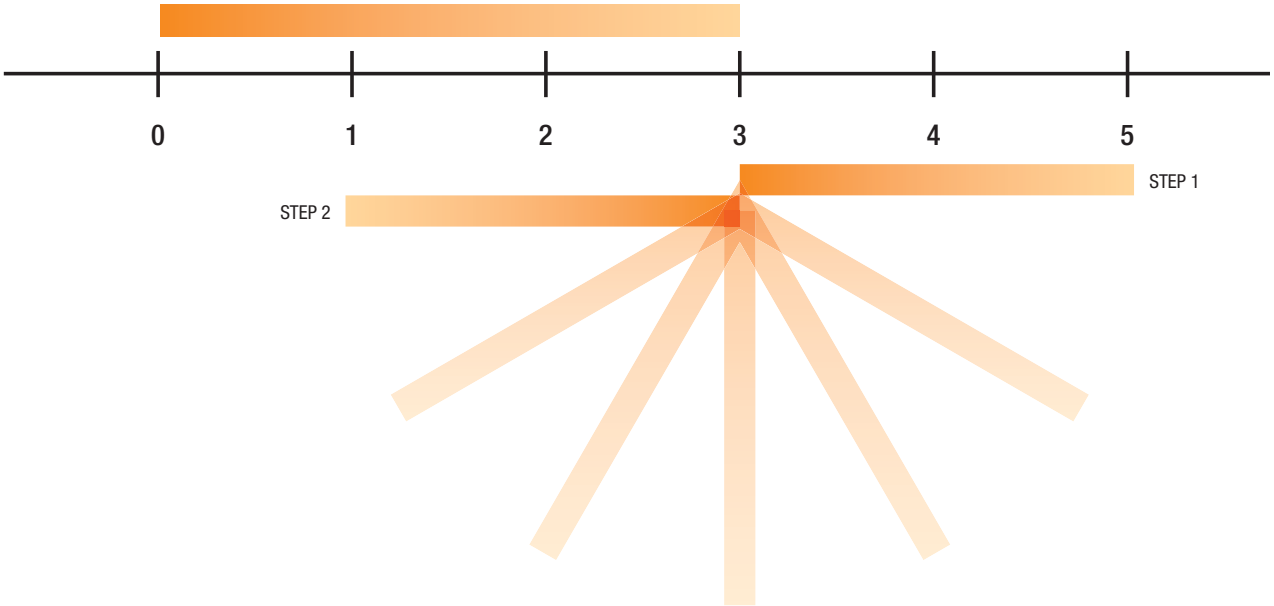
For subtraction, we don't put the bars end-to-end. We put them next to each other, with one end of each matching an end of the other, and find the length of the overhang. In fact, our bars are already next to each other, with their left ends (at 0) matching. Then the overhang is just a bar, going from the number being subtracted, to the one being subtracted from. Its length is the distance between the two numbers. This in fact is a very important interpretation of subtraction. So, it might seem that we automatically have the answer, and don't have to do anything more.

However, although we have a bar whose length is the answer we want, we can't just read off the number that gives its length. We would like to position the bars to create an overhang whose left endpoint is at 0, so that we could read off its length from the right hand endpoint.

There are two ways to do this. One is pretty simple: just slide the shorter bar along the line until its right endpoint matches the right endpoint of the longer bar. Then its left endpoint will mark the end of the overhang, which will start at 0, so this point will be the answer we want. This is shown in the figure below.



This will allow us to read off the answer from the number line. But there is a subtler reason to not be completely happy with this solution. Our bars come with two ends. One end, the left end, is standard, at the origin, and the right end tells us the length. When we do addition on the number line, the left end of the bar that gets moved just matches the right end of the bar that is fixed, and the right end of the bar that gets moved tells us the length of the combined bars. We can preserve these roles of the endpoints by imagining that what we do to line the two bars up takes two steps. First, we slide the shorter bar just as we did in addition, so that its left end lines up with the right end of the fixed bar. Then, we flip the shorter bar over that end, so that what was the right end becomes the left end. We think of the original left end as the beginning, and the other end (the new left end) as the ending end, and these roles are preserved by the flipping process.



These two ways of thinking of the process for subtraction using the number line both give the same answer, and at the moment, the second (more complicated) way may seem just that, more complicated. But it will help us deal with arithmetic of signed numbers in a more uniform way, and that is valuable. We will see how this works in another discussion.

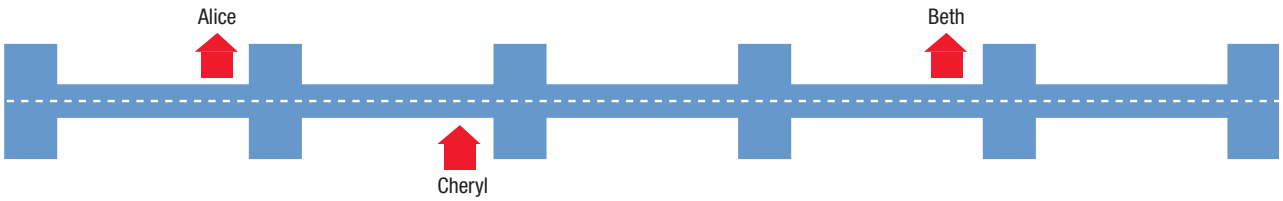
BACK TO TABLE

7. Putting Numbers on the Number Line—Signed (Positive and Negative) Numbers

Think about this word problem:

Alice, Beth, and Cheryl all live on Elm Street. Beth lives 3 blocks from Alice, and Cheryl lives 2 blocks from Beth. How far does Cheryl live from Alice?

What was your answer? Many people will answer "5 blocks" because if you walk 3 blocks from Alice's house to Beth's, and then 2 blocks from Beth's to Cheryl's, you will walk 5 blocks in all, since 3 + 2 = 5. However, Cheryl actually lives only one block from Alice. The picture below shows where the girls live.



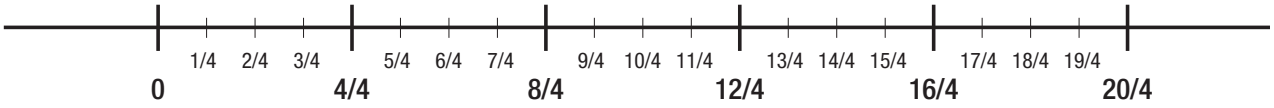
This example shows that, in order to understand location or motion, you need to know not only about distance, but also about direction. This is also necessary for understanding the full number line. The arithmetic version of direction is sign: positive or negative.

Here we will discuss putting both positive and negative numbers on the number line. You should already understand how to place positive numbers. The main emphasis here will be on extending this process to negative numbers.

You have probably seen many pictures of the number line that look something like this.

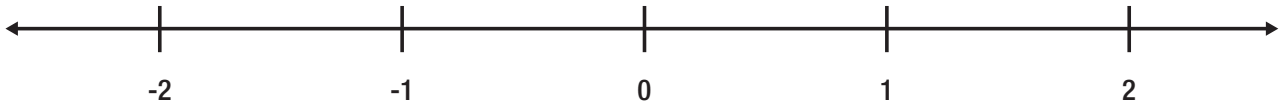
The whole numbers form a regularly spaced set of points marching off to the right from the origin (the location of 0). Each number tells how far the point it labels is from the origin, as a multiple of the unit length, which is the distance from 0 to 1. If we also want to locate some a fraction on the line, for example, 7/4, we:

i) subdivide the unit interval into four equal parts, and
ii) take one of those parts and put points at distances that are multiples of it from 0.
These are 1/4, 2/4, 3/4, etc. So the seventh one is 7/4.



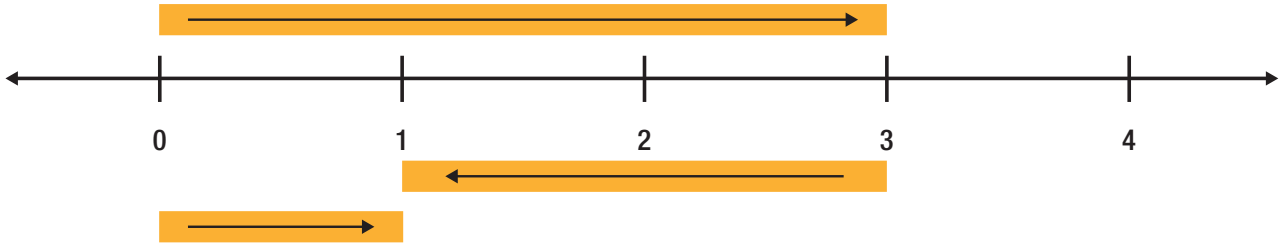
To include negative numbers in this picture, we need to expand our picture to the left, to include the whole line. Although it is common to call the diagrams above “number lines,” in fact, they are only half of the line, the part to the right of the origin. It might be better to call this the “number ray,” but most people just say “number line.”

In any case, the whole number line extends in both directions from the origin. The origin separates the line into two half lines. One side, usually the right side, is where the positive numbers live, and the other side, usually the left, is home for the negative numbers.



Just as when we only looked at the positive half of the line, these numbers are telling you how far the points they label are from the origin. But now that there are two directions to worry about, they are also telling you “in the positive direction.” When we put a minus sign in front of a positive number, we mean “same distance from the origin, but in the negative direction.”

As with the positive numbers, each signed number is connected to an interval, or bar, but now the interval has a direction, given by the sign. The beginning end of the interval is always at 0, and the ending point is at the number. So the positive numbers give intervals that are pointing to the right, and the negative numbers are pointing to the left. You can think of the interval as being equipped with a little arrow that shows its direction. So the path from Alice’s to Beth’s is an interval pointing 3 units to the right (which in this example are blocks), and the path from Beth’s to Cheryl’s is an interval pointing 2 units to the left, and the total result of traveling these two paths is a path 1 unit (= 3 - 2) to the right. The picture below illustrates this.



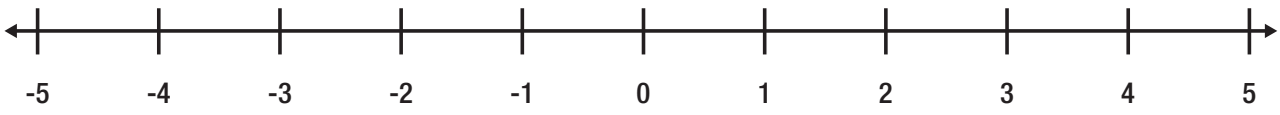
This is how the addition of signed numbers works. We will discuss this in more detail at another time.

BACK TO TABLE

8. Addition on the Number Line—Signed Numbers

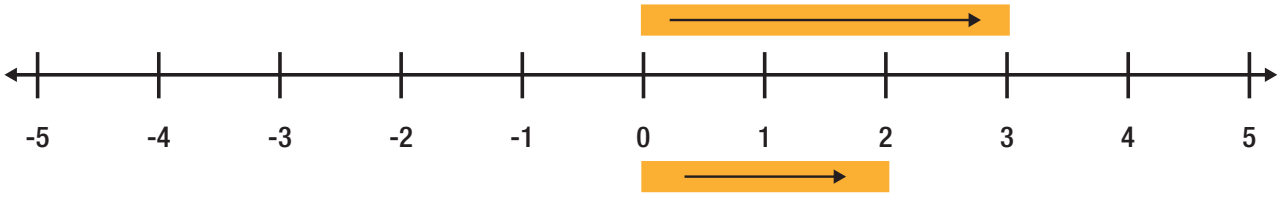
We want to look at how to add signed numbers using the number line. You should already understand how this works with positive numbers. Extending these ideas to signed numbers has the same advantages as with just positive numbers—it doesn't matter if the numbers are integers (that is, signed whole numbers), or rational numbers (signed fractions), or signed decimals—the geometric picture is just the same.

The key thing to understand in adding signed numbers on the number line is the role of **orientation**, or **direction**. When we deal with the whole number line, it is divided into two half lines by 0, the origin.

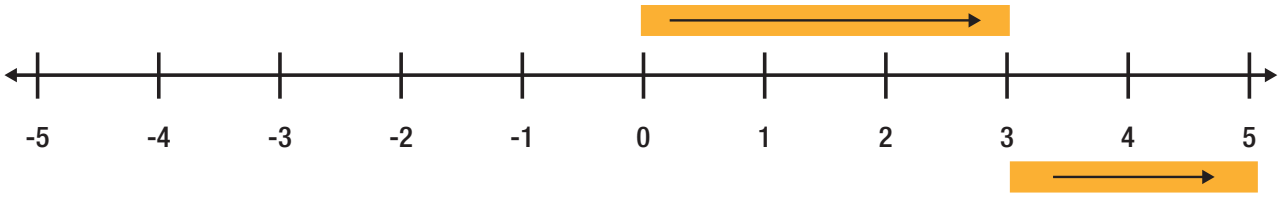


The positive numbers are on one side of 0 (usually the right side), and the negative numbers are on the other. Each number defines a **directed** line segment, going **from** 0 **to** the number. The origin is the starting point of the segment, and the number is the ending point. If we just want to talk about the length of the segment, and not worry about its direction, we use the term **absolute value** of the number.

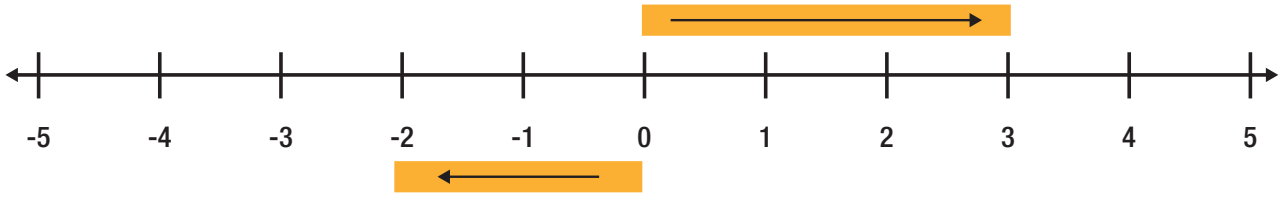
When we use the number line to add positive numbers, we first form the two segments that go from 0 to the numbers. For example, suppose we want to add 3 + 2. The figure below shows the segments corresponding to 3 and 2.

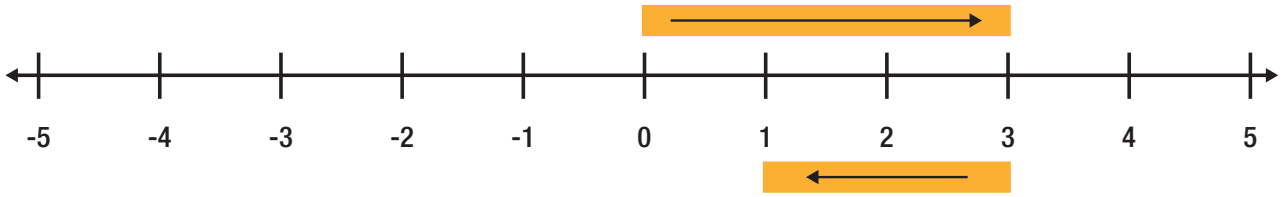


To add, we slide one of the segments so that its beginning end matches with the ending end of the other segment. Then the ending of the segment that was slid is at the sum:



For adding signed numbers, we do the same thing, being sure to keep in mind the direction of each segment. Here is 3 + (-2).



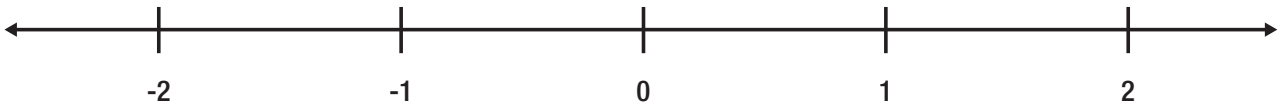


You can see from the picture that the sum is 1. This is the same result as computing 3 minus 2. This is not an accident. In fact, when we include negative numbers in our number system, in some sense, subtraction is not needed any more. Subtraction is just addition of the negative. We will talk about this in more detail in another note.

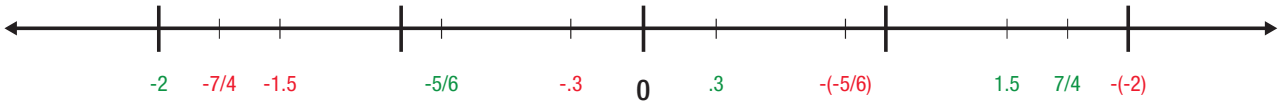
BACK TO TABLE

9. The Negative Numbers and the Number Line

You have learned that addition and subtraction are related from an early age. However, when you learn about signed numbers, it is possible to be much more precise about the relationship between addition and subtraction. The number line is helpful for understanding this relationship. Before trying to deal with this in detail, it is a good idea to understand the concept of the **negative of a number**. To read this, you should already know how to put numbers on the full (two-sided) number line.



For each number on the line, there is another number, at an equal distance from the origin, but on the other side. This number is called the **negative** of the original number. Because of its position on the number line, it is also sometimes called the **opposite** of the original number. Here are a few examples.



We use a minus sign to label the negative of a number. This applies whether the original number is positive, or negative. A key fact to remember that the relationship between the two numbers is mutual: the negative of the negative is the original number. For our examples above, this says that

$$-(-(7/4)) = 7/4 \qquad -(-1.5) = 1.5 \qquad -(-.3) = .3 \qquad -(-(-5/6)) = -(5/6) \qquad -(-(-2)) = -2.$$

If the original number is positive, its negative is negative. That is, its negative is what we usually call a negative number. But if the original number is itself negative, then its negative is positive.

It is important to keep in mind that that the word “negative” gets used in two different ways, although they are related. When we say “the negative,” or “the negative of a number,” we mean the number that is at equal distance from the origin as the opposite number, but on the other side. But when we say just “negative number,” we mean the negative of a positive number, which is a number on the other side of the origin from the positive numbers. “The negative” is a relationship between two numbers. Just plain “negative” tells something about location of the number. It can be confusing, so try to make sure that people understand which option you mean. Some people use “opposite” or “the opposite” when they mean the relationship, and use “negative” only to describe location on the negative half of the number line.

The number line helps us think about the relationship of “the negative,” or “the opposite,” geometrically. If you imagine a mirror located at the origin, then taking a number to its negative amounts to reflecting the number in the mirror. You can also imagine doing this all at once to all numbers, so the whole line gets reflected to itself, with each half line (positive or negative) on one side of the origin going to the other half. Or if you prefer, you can imagine the number line sitting inside a plane, and you can rotate the plane around the origin by 180 degrees, so that when you are done, the line as a whole is in the same place as it was, but the two halves have been exchanged. If you reflect twice, or rotate twice, you get back to where you started. The relationship between a number and its negative is completely symmetric.

This relationship of mutual negatives also applies when we think of numbers in terms of line segments. Remember that each number on the number line tells the length of the line segment between the origin and the number. Also, this line segment has an orientation, or direction, from the origin to the number. The origin is the starting point, and the number is the endpoint. When we reflect the number line over the origin, the line segment from 0 to a given number turns into the line segment from 0 to the negative of the number. The length is the same, but the direction is reversed. The origin or starting point stays where it is, and the ending point is sent to the new ending point. So the orientation of each line segment is reversed at the same time that the number line is reflected.

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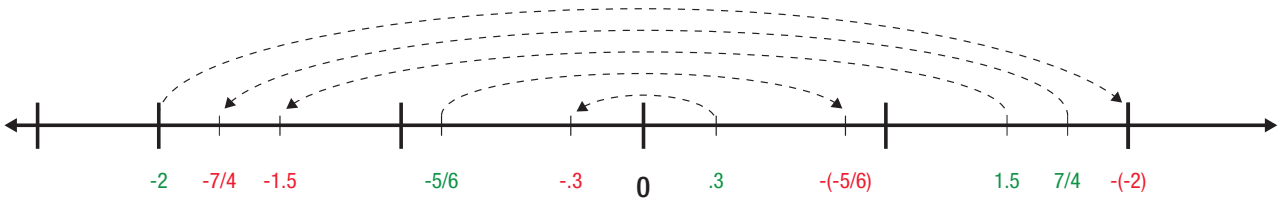
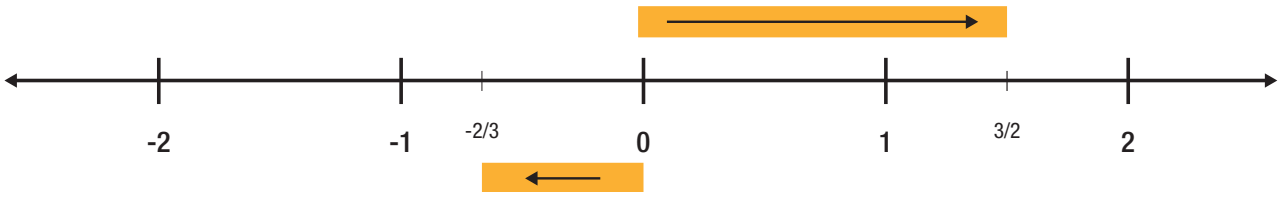
10. Unifying Addition and Subtraction

Addition and subtraction are linked from the beginning of the study of arithmetic. But some students never learn just how closely they are connected. In some sense, subtraction is (almost) included in addition. To be more precise, if you understand the idea of the negative of a number, you can say: subtraction (of a number) is the same as addition of its negative. Let's look at how this works in an example.

The negative of a number can be understood geometrically as a reflection of the number across the origin of the number line (or you can imagine the number line as a very thin, very straight stick, and rotate it 180 degrees around the origin). The negative of a positive number is a negative number, and the negative of a negative number is positive. Taking the negative twice gets you back to your original number.

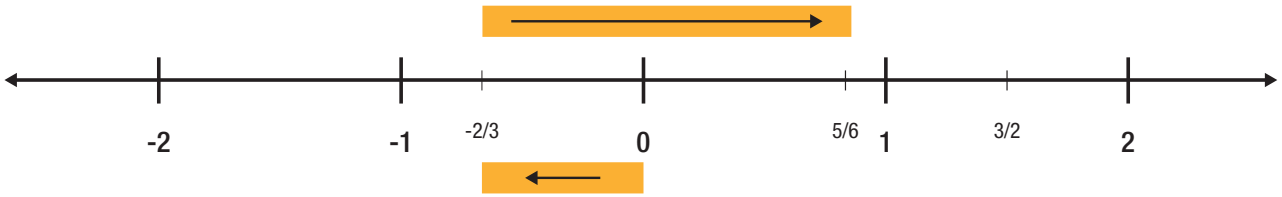


Remember how we add numbers on the number line. We will do the example of 3/2 plus -2/3. Each number is the end of a line segment that goes from the origin to the number. The segment is oriented: the origin is its starting end, and the number is its ending end. Here is the picture of the segments for 3/2 and -2/3.



To do the addition, we slide one of the segments until its beginning end matches the ending end of the other segment. Then the ending end of the segment that moved is at the sum of the two numbers.

Or:



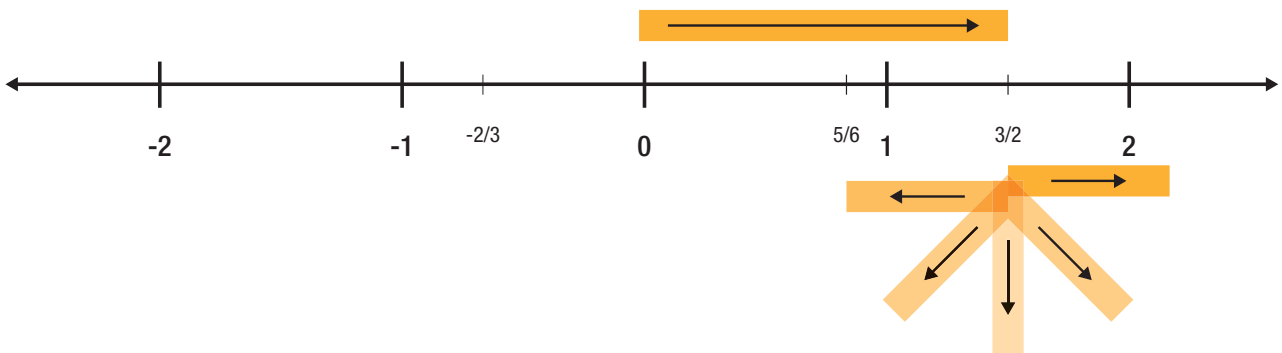
As these pictures show, 3/2 + (-(2/3) = 5/6. To compute this symbolically, we would use the fact that both halves and thirds can be expressed in terms of sixths: 1/2 = 3/6, and 1/3 = 2/6. So

$$3/2 = 3\times(3/6) = 9/6, \text{ and } -(2/3) = -(2\times(2/6)) = -(4/6) = (-4/6), \text{ and}$$

$$3/2 + (-2/3) = 9/6 + (-4/6) = (9 + (-4))/6 = 5/6.$$

The pictures also show that it does not matter which segment we keep fixed and which one we slide, the answer will be the same. This is the Commutative Rule for addition.

Suppose instead, we want to subtract 2/3 from 3/2. We focus on the directed segments from 0 to 3/2 and from 0 to 2/3. First, similarly to addition, we slide the segment of the number we want to subtract (for us, 2/3) so that its beginning endpoint (the one at 0) is below the ending endpoint of the other one (in this case, 3/2). Now the ending endpoint of the segment that we slid is at 3/2 + 2/3.



But we want to subtract the 2/3, not add it. To do that, there is one more step: we reverse the direction of the segment for 2/3, keeping its starting end fixed. Now it is pointing to the left rather than the right, and its ending endpoint is at 3/2 - 2/3.

If we look at the figure that gives us 3/2 - 2/3, we can see that it is in fact the same as the figure that gives us 3/2 + (-2/3). You should try to convince yourself that this will always work, no matter what the signs are of the numbers to be added. It is always true that subtracting a number gives the same result as adding its negative.

A very important special case of the fact stated in the above paragraph is that if you take the sum of a number and its negative, you always get 0. Adding the negative of a number to that same number just cancels both out. This is a key property of the negative, and it is sometimes expressed by using an equation with a variable. If x stands for any number, then $-x$ is used to stand for its negative. If we use these symbols, then the key relationship between a number and its negative can be expressed by saying that

$$x + (-x) = (-x) + x = 0.$$

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11. Basics of the Place Value System

The place value notation that we use to express whole numbers is an extremely clever and efficient method. It took thousands of years and several stages of improvement before it reached the sophisticated form that we use today. This form uses several ideas that are today seen as being part of algebra. Base ten place value notation is probably the most widely used system in the world today. People speak different languages in the US and Canada, in South America, in Europe, Africa, India, and China, but everybody uses the base ten place value system.

All students work with the base ten system in every grade, starting in kindergarten and all the way through high school. But many students do not come to understand the basic ideas behind place value, and this makes it more difficult to deal with arithmetic. For some students, the problems start already with addition and subtraction of numbers with three digits or more. Let's spend some time looking carefully at what place value notation is telling us.

Take a base ten number, for example 352. What does this mean? First, each digit stands for a piece of the number, and the whole number is a sum of the pieces:

$$352 = 300 + 50 + 2.$$

It is useful to have a name for the pieces. The name we will use is **place value parts**. So 352 has three place value parts: 300 and 50 and 2. There is one place value part for each digit of the number.

Each of the place value parts is clearly a very special kind of number. It has only one non-zero digit. What are the digits telling us? They are counting units of different sizes. 300 means "three hundreds," and 50 means "five tens," and 2 means "2 ones." The idea behind this is that when numbers get large, it is hard to tell what they mean unless you chunk them into pieces of agreed-upon sizes. (Try counting these commas. ,,, How many are there? And this number is a lot less than even 100! How long would it take you to count them if there were 352? And how sure would you be that you were right?)

The base ten system uses standard sizes that go up by multiples of 10. So the unit of the second place is ten, and the unit of the third place is 100, which is ten times ten. If we had a digit in the fourth place, it would stand for one thousand, which is ten times one hundred. This relationship between the places continues: each digit stands for the number of units of a size that is ten times the size of the next place to the right. We will call these numbers: 1, 10, 100, 1000, 10000, and so on, the **base ten units**.

To summarize, the digits of a number are telling us how many of each base ten unit we need to make that number. So, each place value part is actually a product of the base ten unit for that place, and the digit for that place:

$$300 = 3 \times 100, \quad \text{and} \quad 50 = 5 \times 10, \quad \text{and} \quad 2 = 2 \times 1.$$

Because each base ten unit equals 10 of the next smaller unit, we never need or use more than 9 = 10 - 1 of any single unit. If in the process of some calculation, we get ten (or more) of some unit, we convert ten of it to one of the next larger unit. So having 10 digits, that go from 0 up to 9, is all we need to write base ten numbers.

In this system, zero plays an absolutely critical and essential role. When there is a zero in a place, it is telling us that we do not need any of that particular size unit. You might think that since it doesn't add anything to the number, we could just leave it out. But if we did that, we would mess up the system of places. Take the number 504. It doesn't use any tens. But it we leave out the 0, we just get 54, which is a very different number than 504! In fact, it took a long time for the idea of zero to be invented. It was the crowning achievement that allowed place value notation to work.

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12. The Size of Base Ten Numbers

One of the nice features of our base ten place value system for writing numbers is that it makes it easy to understand roughly how big numbers are, and it also makes comparing numbers very easy.

You should already understand that a base ten number is a sum of parts, called its place value parts. This is expressed by expanded form. For example, the expanded form of 7,414 is

$$7{,}414 = 7{,}000 + 400 + 10 + 4.$$

The main point we want to emphasize here is that, of these place value parts, the parts get smaller as you move to the right in the digits. In fact, they get smaller rapidly. The 7,000 is the largest part. It is larger than all the other parts combined. Then the next largest part is the 400, and it is larger than the sum of the rest of the parts. And then 10 is the third largest part, and the second 4 is the fourth largest, which is also the smallest in this example.

These relationships are always true, even if the digits themselves get larger as you go to the right in the places. Take a number like 1,369. The expanded form is

$$1{,}369 = 1{,}000 + 300 + 60 + 9.$$

The largest of the parts is 1,000, and it is larger than the last three parts combined. Then the next largest part is 300, then 60, then 9. Even in a number like 19 = 10 + 9, the 10 is larger than the 9.

To be completely sure about this, it may help to think about how we count. If we start out counting, at first, we only use one digit: 1, 2, 3, . . . , 9. Then after 9, we get to ten, which we write 10, which means 1 ten and zero ones. What the base ten notation does is collect the ten ones into one ten—the first base ten unit (after 1, the basic unit). Then we continue with 11, which means "one ten and one one," and 12 (which means "one ten and two ones"), up to 19 (one ten and nine ones). Then if we add another one, we fill up a second ten, so we write it as 20 (two tens and zero extra ones). We then continue with 21 (two tens and one one), up to 29 (two tens and nine ones). Adding another one again fills up a ten, so now we have three tens, which we write as 30.

We continue like this, one at a time, and each time we get to 10 ones, we collect them into a ten, and start counting the extra ones over again from zero. This goes on until we get to 99. Now if we add one more one, we will again have ten ones, which we again collect into a ten, which, together with the nine tens we already have, makes ten tens. This is the next base ten unit, 100, or 10 tens, or 10 × 10. Then the counting starts all over again, and we run through 101, 102, . . . , 109, 110, . . . , 120, ... , 199, all of which stand for "one hundred, plus

some number less than 100." When we add one more to 199, the extra 1 combines with the 99 to make another hundred, so now we have two hundreds, which in symbols is 200. We continue counting like this, creating another ten each time we get to ten ones, and creating another hundred each time we get to ten tens, until we reach 999. Now when we add one more one, we make another 10, which combines with the 9 tens that are there to make ten tens, which we consolidate into another hundred, which makes 10 hundreds, and this is the next higher base ten unit, namely 1,000. And so on.

Looking at this process, we see that we first use two digits when we run out of single digits, and in particular 10 is greater than 9. Then we first use three digits after we have used the largest possible combination of two digits, namely 99, so 100 is greater than 99. And we first use four digits after we get to 999, which is the largest possible combination of three digits. From the opposite point of view, 10 is the smallest two-digit number, and 100 is the smallest possible three-digit number, and 1,000 is the smallest possible four-digit number, and so forth.

This means that every two-digit number is larger than every one-digit number, and every three-digit number is larger than every two-digit number, and so forth. This may seem like a pretty simple conclusion, but it can help us find an easy way to compare any two numbers. That will be the subject of another discussion.

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13. Comparing Base Ten Numbers

Our base ten system for writing numbers is both very compact and very powerful. It helps us with expressing numbers, with computing with them, and with approximation. One of the simplest parts of approximation is telling which of two base ten numbers is larger. There is a simple rule for this.

To understand the rule, you should already know that place value notation is a compact way of expressing a number as a sum of special numbers, its place value parts. Place value parts are numbers with only one nonzero digit. For example,

$$2{,}635 = 2{,}000 + 600 + 30 + 5$$

is the expanded form of 2,635, breaking it up into its place value parts.

There are intermediate ways of breaking up a number, which group several place value parts together. For example, you can say

$$2{,}635 = 2{,}000 + 635, \quad \text{or} \quad 2{,}635 = 2{,}600 + 35, \quad \text{or} \quad 2{,}635 = 2{,}600 + 30 + 5.$$

There are lots of options for breaking up a number into combinations of its place value parts, depending on which one is most helpful for what you want to do.

With regard to comparison, you should already understand that if two base ten numbers have different numbers of digits, then the one with more digits is larger. This is true regardless of what the digits are. In particular, even a base ten unit, which is the smallest base ten number with that many digits, is larger than any number with fewer digits. This simple fact allows us to figure out which of any two base ten numbers is the larger one.

Let's look at some examples. First, take the case when two numbers have the same number of digits, but the leading digit (the one on the left) of one is larger than the leading digit of the other. Then, the one with the larger leading digit is larger. Here is an example: compare 276 and 324. We claim that 276 is less than 324. We write this as $276 < 324$. Here is the argument that this is true:

$$276 = 200 + 76 < 200 + 100 = 300 < 300 + 24 = 324.$$

In this argument, we have used a principle that should seem reasonable:

> **If a first number is less than a second number, and we add the same number to both of them, then the first sum is less than the second sum.**

Above, for the first inequality, we have added 200 to 76 and to 100. For the second inequality, we have added 300 to 0 and to 24.

Another principle we used is basic for dealing with inequalities:

> **If a first number is less than a second number, and if the second number is less than a third number, then the first number is less than the third number.**

This property is used so much, it has a name: it is called **transitivity**.

Now let's see a trickier example of how these ideas let us decide which of two numbers is larger. We will take 57,276 and 57,324. Since the two digits on the left of both numbers are the same, let's decompose them into parts by these two digits. We get

$$57{,}276 = 57{,}000 + 276, \quad \text{and}$$

$$57{,}324 = 57{,}000 + 324.$$

The biggest parts of both numbers (the 57,000 parts) are exactly the same, so they just balance each other. The parts that are different (the 276 and 324) determine which number is larger. And since 324 has the largest place value part between these two, it is the larger one. We have already seen how this works. Now if we add 57,000 to both numbers, we get

$$57{,}000 + 276 < 57{,}000 + 300 < 57{,}000 + 324.$$

Then putting the numbers back together, we finally get that

$$57{,}276 < 57{,}300 < 57{,}324.$$

This argument works every time. Here is another example, without as much commentary. Compare 8,291 and 8,278. We break the numbers up:

$$8{,}291 = 8{,}200 + 91 \quad \text{and} \quad 8{,}278 = 8{,}200 + 78.$$

Then we compare, looking at the parts that differ:

$$8{,}200 + 91 > 8{,}200 + 90 > 8{,}200 + 78.$$

Therefore, 8,291 > 8,278.

Perhaps you have already figured out the general rule for comparing, but we will state it formally:

i) If two base ten whole numbers have different numbers of digits, then the one with more digits is the larger number.

ii) To compare two base ten whole numbers with the same number of digits, look for the largest (leftmost) place where the numbers have different digits. Then the number with the larger digit in that place is the larger number.

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14. The Size of Base Ten Numbers and the Number Line

For this discussion, you should already know that a base ten number breaks up into a sum of place value parts that are the products of the digits of the number with the corresponding base ten units. For example,

$$7{,}944 = 7{,}000 + 900 + 40 + 4 = 7 \times 1{,}000 \quad + \quad 9 \times 100 \quad + \quad 4 \times 10 \quad + \quad 4 \times 1.$$

You also should have learned that the largest place value part is the one corresponding to the leading digit—the one on the left. This is larger than all the other place value parts combined. As you move to the right in the places, the place value parts get smaller and smaller. In fact, they get smaller very fast, because each base ten unit is ten times as large as the next smaller one. This feature has the advantage that it allows us to express very large numbers very compactly, but it has the disadvantage that it is hard to comprehend how large the numbers, that we write so easily, really are. Looking at the work of placing these numbers on the number line can help + visualize and compare the contributions of the different place value parts to the size of the number.

The typical picture of the (one-sided) number line looks something like this:

But of course, larger numbers like 7,944 will not come anywhere near fitting on a line with this scale, that also fits in the room where you are! To think about placing 7,944 on a number line, the scale has to be very large, so the corresponding unit length will be very small.

The smallest base ten unit that is larger than 7,944 is 10,000. So let's imagine a number line that goes from 0 to 10,000, but is small enough for us to work with. It will look something like this:

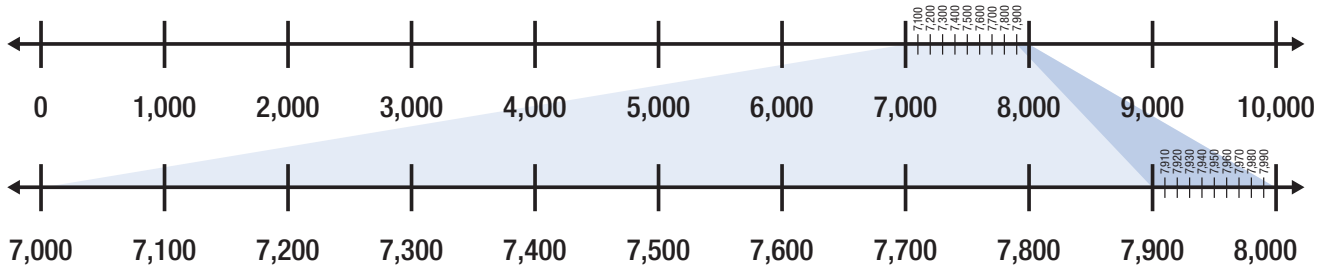
Where does 7,944 go on this line? Since 7,000 < 7,944 < 8,000, it will go somewhere in the eighth interval, between the locations of 7,000 and 8,000.

To locate 7,944 more accurately, we subdivide the interval between 7,000 and 8,000 into ten equal subintervals. The numbers separating these smaller intervals will be 7100, 7200, 7300, up to 7,900.

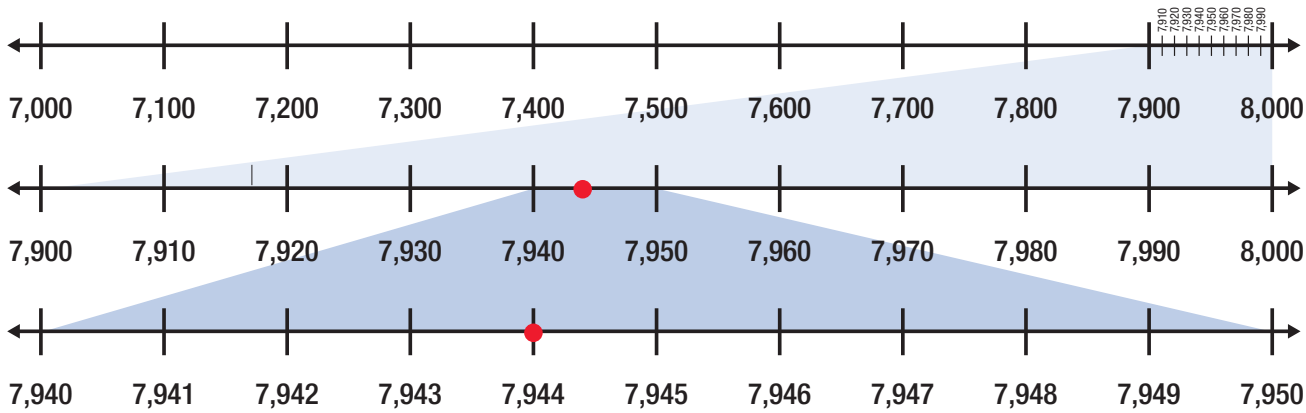
Since our number 7,944 satisfies 7,900 < 7,944 < 8,000, it will lie in the last (the tenth) of the smaller subintervals.

To locate our number more accurately, we should repeat the subdivision process we have just gone through. We should subdivide the interval between 7,900 and 8,000 into ten equal subintervals. Then 7,944 will go in the fifth of these subintervals.

But, these subintervals will be rather small! Even the intervals of length 100 are small. You need to have pretty good eyes to read the labels on them. The intervals representing length 10 would be very difficult to even see. They would be thinner than the lines we have been using to separate intervals! We would not be able to draw them. To make the process more visible, we imagine "blowing up" the interval between 7,900 and 8,000, so we can see the smaller intervals better. First, we blow up the interval between 7,000 and 8,000:



Now we can locate our number between 7940 and 7950—it is somewhere in the fifth subinterval of the tenth interval. But again, the scale we are working at is too small to show the detail that we want. So we blow it up, twice more.



That is how to find the location of a base ten number: starting from the largest place value piece, you can locate an interval where the number must be. Then subdividing this interval into ten pieces, by looking at the next place value piece, you can tell which of the ten subintervals the number will be in. Then you repeat this with each successive place value piece: subdivide the interval where you know your number is into ten equal subintervals, and use the next place

value piece to tell which of these ten smaller intervals your number is in. Repeat and repeat, until you get to the last place value piece (for whole numbers, this will be a single digit).

An important realization that comes out of this process is that in comparison to the largest two or three place value pieces, the smaller ones contribute very little to the size of the number. For many purposes, and especially if the number comes from measurements, the smaller place value pieces almost don't matter. (Or even worse, they may be wrong! But this is a complicated issue that needs a separate discussion.)

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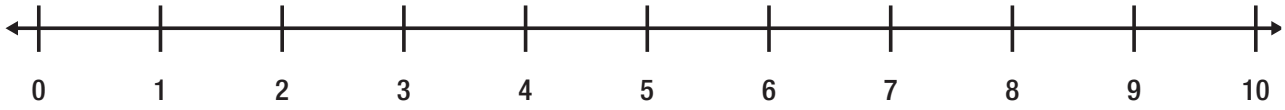
15. Putting Decimals on the Number Line

To read this, you should already know about how to place whole numbers on the number line. For small whole numbers (say, under 100), it is fairly straightforward, but for larger numbers, you need a more systematic strategy based on working with the place value parts from largest to smaller, and careful attention must be paid to issues of scale.

Here we will study putting decimal fractions on the number line. The main point is that the process is quite parallel to putting multidigit whole numbers on the line. The overall scale is different, but the relative sizes of the scales involved is the same as for whole numbers.

Let's look at an example. We will use a decimal whose whole number part is a single digit. That way, we will get to the fractional part right away. Let's try placing 7.944. Here is the number line as you usually see it, from 0 to 10.



Since the whole number part of our number is 7, we know that it will go somewhere between 7 and 8. To see where, we want to break up the number into its place value parts. Decimal fractions have an expanded form as a sum of place value parts, just as whole numbers do. The expanded form of 7.944 is

$$7.944 = 7 + .9 + .04 + .004 = 7 \times 1 + 9 \times .1 + 4 \times .01 + 4 \times .001.$$

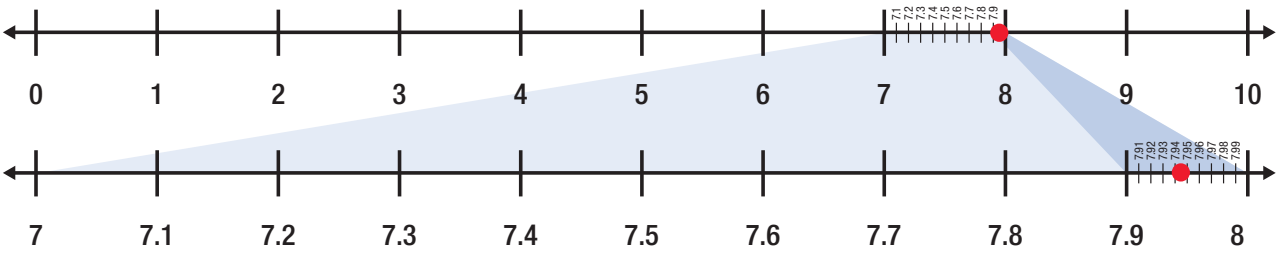
The numbers $.1 = \frac{1}{10}$, $.01 = \frac{1}{100}$ and $.001 = \frac{1}{1000}$ are the first three fractional base ten units. The fractional base ten units maintain the same multiplicative relationship to each other as do the whole number base ten units: each one is ten times as large as the next smaller unit, and it one-tenth as large as the next larger unit.

Because of these relationships between successive base ten units, the process of locating a decimal fraction on the number line is quite parallel to locating whole numbers.

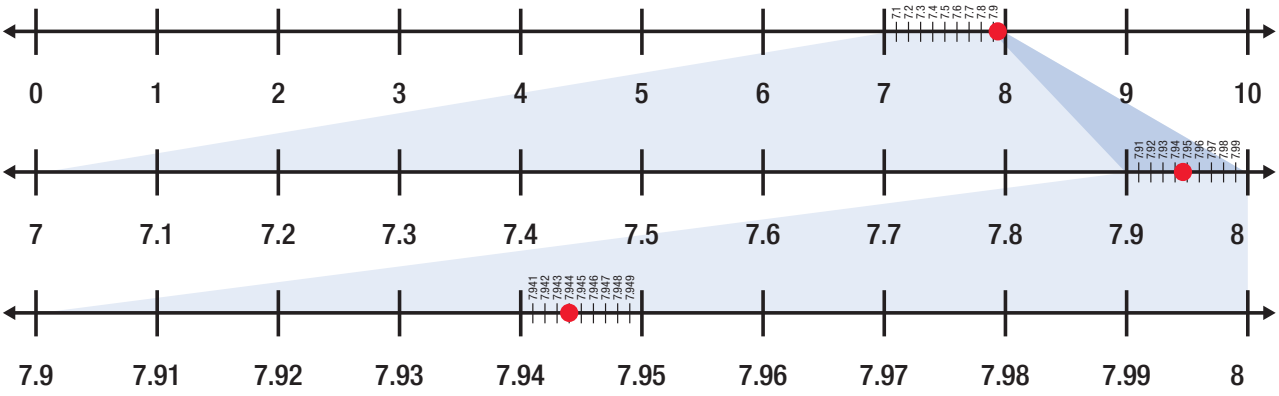
Since the whole number part of 7.944 is 7, the number will go somewhere between 7 and 8 on our number line. To locate it more accurately, we subdivide the unit interval between 7 and 8 into ten equal parts.

Since the coefficient of .1 in our number is 9, it will go somewhere between 7.9 and 8.

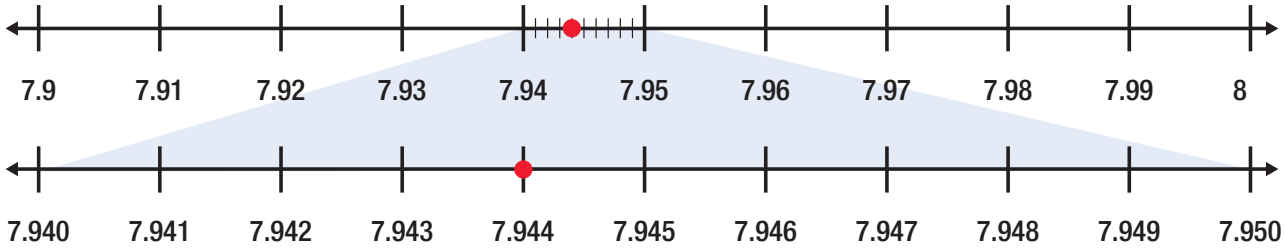
To continue homing in on its exact location, we should subdivide that interval into ten equal pieces. However, these pieces will be quite small, almost too small to see! In order to see the location more exactly, we blow up this interval.



Since the next digit (the multiple of .01) in our number is 4, our number will go in the fifth subinterval, between 7.94 and 7.95.



To finally locate the number exactly, we again blow up the relevant interval.



Since the digit multiplying .001 is 4, our number goes at the end of the fourth/start of the fifth subinterval. We are done.

As when locating a whole number, we see that the sizes of the successive place value parts decrease rapidly as we move to the right in the base ten expression of the number. In fact, although the absolute scale of various numbers may be quite different, the relationship between sizes of successive places is the same, and as this example and the parallel whole number example illustrate, after the first two or three largest parts, the contribution of the remaining place value parts make comparatively little difference in the size of the number. It is also worthwhile to study this phenomenon from a numerical viewpoint. This will be the topic of another discussion.

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16. Approximating Numbers by Rounding

The base ten place value system is a wonderful method for representing whole numbers, and also for computing with them. However, as the number of digits in a number increases, it does become increasingly tedious to deal with: to read, remember, and compute with it. Fortunately, the base ten system is also very efficient at approximating numbers. Any number can be pretty well approximated by numbers with only a few place value parts. This approximation process is known as **rounding**.

Very likely, you have already had some experience with rounding, but we will review how it is done. There are actually three types of rounding that are often discussed: rounding up, rounding down, and rounding-to-the-nearest. Rounding-to-the-nearest is kind of a fussy procedure that may not be presented consistently from textbook to textbook, and it is only needed in specialized situations. We will ignore it.

Rounding down is the simplest kind of rounding. It is done by choosing a place value part, and dropping all the parts smaller than that one—just replacing them with 0. For example, 3,614 rounded down to the hundreds is 3,600. And 3,614 rounded down to the tens is 3,610.

Rounding up is also done by choosing a place value part, and replacing the smaller parts with one copy of the base ten unit of the chosen size. This is then added to the place value part that is already there, and so increases the value of that digit by 1. Our number 3,614 rounded up to the hundreds place is 3,700. In more detail, the process is

$$3{,}614 = 3{,}600 + 14 < 3{,}600 + 100 = 3{,}700.$$

If instead we round 3,614 up in the tens place, the result is 3,620.

Rounding up is slightly trickier than rounding down, because if the digit in the place where you are rounding is a 9, then adding 1 to it makes a 10, and turns it into the base ten unit of the next larger size. Sometimes, this can involve changing several digits. For example, 2,997 rounded up to the next hundred is 2,900 + 100 = 3,000, and 2,997 rounded up to the next 10 is 2,990 + 10 = 3,000.

The question we want to look at is: how much does rounding change a number?

The first answer to that is very easy: it changes the number by, at most, one copy of the base ten unit for the place where rounding is done.

A more refined statement of the change created by rounding is that the difference between rounding up and rounding down at a given place is exactly one base ten unit corresponding to that place. For our examples above, this amounts to the inequalities

$$3{,}600 < 3{,}614 < 3{,}700, \qquad \text{and}$$

$$3{,}610 < 3{,}614 < 3{,}620.$$

So it is easy to see the most that rounding can change a number in absolute size. For either rounding up or rounding down, the change is never more than one base ten unit of the place where the rounding is done.

But a more significant question is, how much does rounding change the number **in comparison to the size of the number**? This is a little trickier to answer, but also much more relevant to many situations where we use numbers. If someone owes you $378, and pays you $300, you probably don't want to forget about the missing $78. But if someone owes you $10,378, and pays you $10,300, you are still missing $78, but you will probably feel you have "almost" been paid back.

To compare the difference between the rounded amount and the actual amount, we should divide one by the other. In our example above, with rounding down, this would be

$$\frac{(3614 - 3600)}{3614} = \frac{14}{3614}$$

However, we would like to take the point of view that we don't know exactly what the actual amount is, only that when rounded down to the hundreds, it is 3,600. It could be any number between 3,600 and 3,700 (well, 3,699). In other words, we are imagining that we are dealing with a number 3,6??, and we are not too sure what the digits ?? are. But we do know that

$$0 < ?? < 100.$$

If we put these bounds on the last two digits, then we can say

$$\frac{(36?? - 3600)}{36??} = \frac{??}{36??} < \frac{100}{36??} < \frac{100}{3600} < \frac{1}{36}$$

In the first inequality above, we use the fact that if you increase the numerator of a fraction, you increase the fraction. In the second inequality, we use the fact that if you decrease the denominator of a fraction, you increase the fraction. We hope you are comfortable with these facts.

At any rate, our conclusion is that if we round any number between 3,600 and 3,700 down or up, the error we make compared with the original number is always less than $\frac{1}{36}$. This is much simpler than an exact answer, and we can see that it is fairly small. As a decimal, it is less than .03.

We can continue this line of reasoning and get estimates for rounding error that depend only on the unrounded places, or even only on the number of unrounded places. That will require another discussion.

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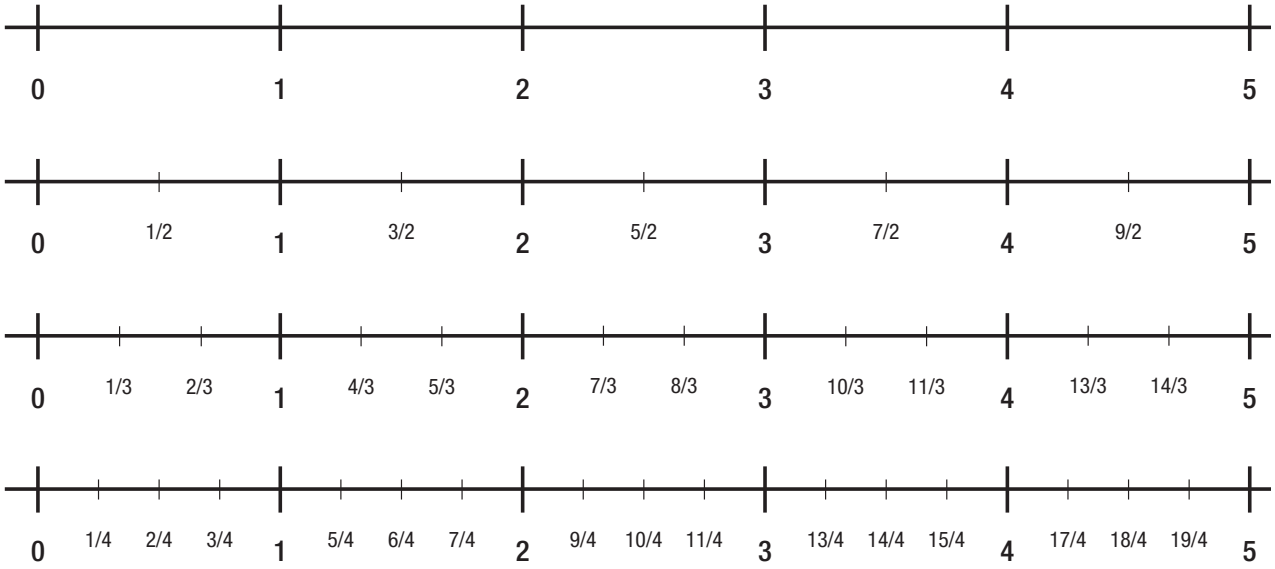
17. The Number Line and the Size of Fractions

Fractions cause lots of problems for lots of students. One obstacle for many students is figuring out how large a fraction is. This is related to the task of comparing fractions—deciding which one of two fractions is the larger one. Thinking about putting fractions on the number line can help with these problems. Since this is a topic that mainly concerns positive fractions, we will only look at the positive part of the number line.

You should already understand the basics of putting fractions on the positive number line. If you fix a denominator, and then look at all possible numerators, you get a nice, regularly spaced system of points. It looks a lot like the whole numbers, but the spacing is closer together. As examples, look at the systems of whole numbers, and multiples of $\frac{1}{2}$, and of $\frac{1}{3}$, and of $\frac{1}{4}$.

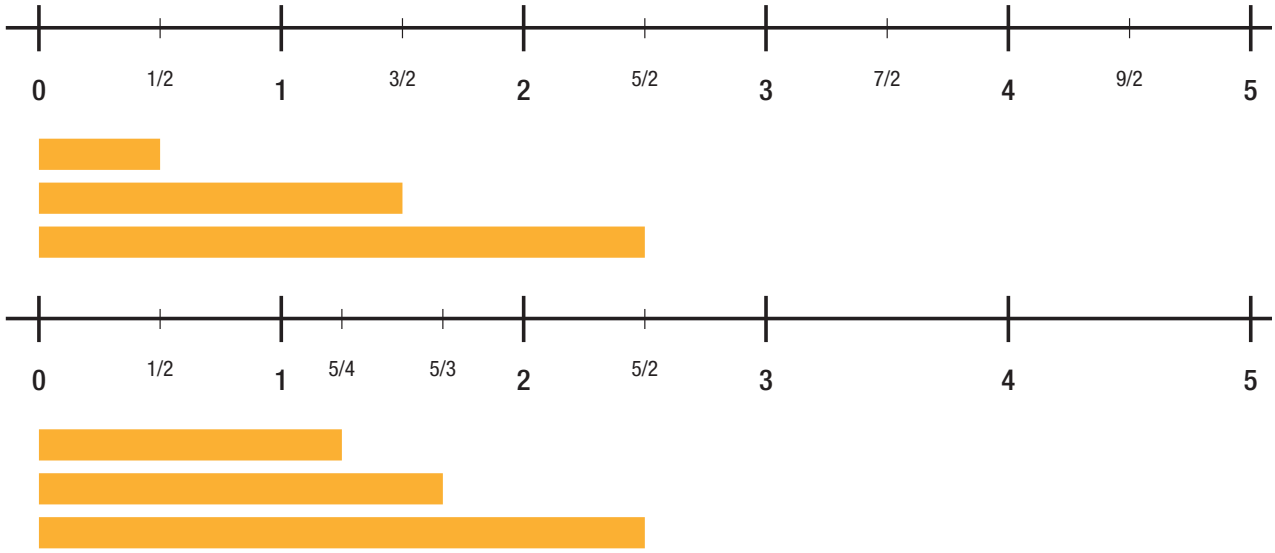


The first thing to ealize (and it should be clear from looking at these pictures) is that if the numerator of a fraction increases while the denominator stays fixed, then the fraction increases. For example,

$$\frac{1}{2} < \frac{3}{2} < \frac{5}{2}.$$

But, if the numerator stays fixed and the denominator increases, then the fraction decreases:

$$\frac{5}{2} > \frac{5}{3} > \frac{5}{4}.$$

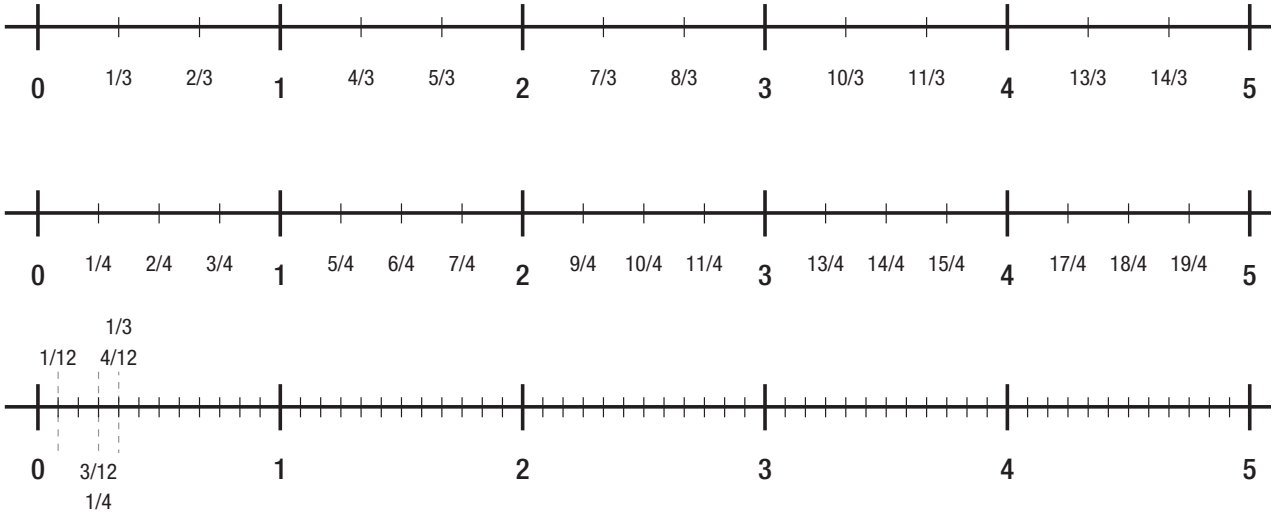


This contrary behavior can be confusing, and it is one of the reasons that comparing fractions is tricky. Increasing the numerator makes a fraction bigger, but then increasing the denominator makes it smaller, and you have to figure out which has more effect. But, if you can understand and remember these two relationships, when only the top or only the bottom changes, they will let you compare fractions with equal numerators or equal denominators. Picturing the number line may help.

They can also help you compare fractions beyond those two very special cases. They also let you compare when the changes in both numerator and denominator have the same effect. So if you start with some fraction, and increase its numerator, and then decrease its denominator, you will increase the fraction. This idea lets you decide on examples like these:

$$\frac{5}{8} < \frac{6}{7} \quad \text{since} \quad \frac{5}{8} < \frac{6}{8} < \frac{6}{7}, \quad \text{and} \quad \frac{7}{12} < \frac{8}{10} \quad .$$

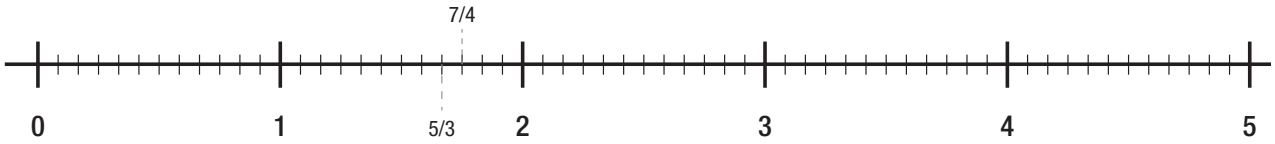
Of course, there is a general standard recipe for deciding which of two fractions is larger. This is to express both fractions as equivalent fractions with the same denominator. Then you can just compare the numerators. Another fact that you should know, and which can be explained nicely with the number line, is that any two fractions can be put in a single system of fractions with the same denominator. One denominator that always works is the product of the two original denominators. For example, the system of thirds and the system of fourths both fit inside the system of twelfths, since 12 = 3×4.



So if we want to compare, say 7/4 with 5/3, we can calculate that

$$7/4 = (7\times3)/(4\times3) = 21/12, \qquad \text{and} \qquad 5/3 = (5\times4)/(3\times4) = 20/12,$$

and then we can conclude that $7/4 > 5/3$, since $21 > 20$. But this computation also helps you see why this conclusion is tricky—since 12 is a not such a small denominator, and the two fractions only differ by 1/12, they are not very far apart. The precise computation is helping us make a fine distinction.



This is kind of a laborious process, since it involves multiplying the numerator of each fraction with the denominator of the other one, so it is nice to have tricks that, when they work, help you avoid all that calculation. But this method will always work, and the number line (by letting us see how the systems of fractions with two given denominators fit inside a single system with denominator equal to the product of the two original ones) can help us understand why it works.

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18. What Are Functions?

Functions play a huge role in mathematics, but they are often hidden. As a student, you meet specific functions very early in the study of mathematics, and many, many of them over time, but you may not hear the word "function," or be formally introduced to the concept of function, until much later—in an algebra course in high school, slightly earlier for a few, maybe not at all. Many of the functions that you work with are never specifically named, and the full idea of what a "function" means is never revealed to many students.

The idea of function comes out of our efforts to deal with quantities that change. We are interested in some quantity, like the temperature, or the length of the day (from sunrise to sunset), or the distance between us and something (a city, if we are on a trip, or a car that is moving toward us, or another player when we play basketball). The temperature where we are changes continually, from morning to noon to night, and tends to be different at different times of year. The length of the day is different at different times of year. To refer to these changes, we say: "temperature is a function of time," and "the length of the day is a function of the time of year," and so forth.

So what is a function? A function is a kind of correspondence, a way of assigning something to something. In the two examples above, a temperature is assigned to a given time, or the length of a day is assigned to the day.

Most functions that get studied in school are numerical functions. A numerical function is a correspondence between numbers. A correspondence between quantities, for example, time and temperature, can be interpreted as a correspondence between numbers, if a unit of measurement is chosen for each quantity, so that the sizes of the quantities can be expressed as multiples of the unit. So many "real-world" functions can be expressed by means of numerical functions. It can be thought of as a recipe for taking a number and producing another number (sometimes the numbers allowed to be used are restricted). The function is the correspondence between the original number and the result of applying the recipe to the original number. Some people also talk about "function machines" that take a number as an input and do something to turn it into an output.

The first family of functions that many students see are **linear** functions. If we take a number x, then a linear function of x is described by an expression like $2x + 3$ or $1 - 3x$. The expression, say $2x + 3$, is understood to be the recipe or rule that takes a number (which is the x), then multiplies it by 2, then adds 3 to that product. The function is the **correspondence** between the original number x (whatever it is), and the ending number, $2x + 3$. For example, 1 goes to 5, and 4 goes to 11. Since there are many possible choices for the number x, it is called a **variable**. If we are being careful, we should precisely describe the collection of allowable values of x, but usually this is left somewhat vague. (Sometimes, if

someone wants to talk about some linear function, but not a specific one, they may write "$ax + b$." What this means is that some numbers a and b are assumed to have been chosen, but at the moment, we do not want to say exactly what they are, and then a linear function of the variable x is determined by the process of taking x, multiplying it by a, and then adding b to the result. In a context like this, the numbers a and b are often called **parameters**.)

After linear functions, the next notable family is "quadratic functions," which are designated by expressions that involve x^2 as well as x and fixed number (known as "the constant"). Examples are just x^2 itself, which in some sense is the basic model for a quadratic function, and also more complicated expressions: $2x^2 - 3x + \frac{13}{4}$. Remember that the function is the **correspondence** between a starting number, called x, and the ending number $2x^2 - 3x + \frac{13}{4}$. This expression is how we communicate the rule:

Take a number x, square it and multiply that by 2; then multiply x by -3, and add the two products; finally add $\frac{13}{4}$ to the number you have just computed; this is $2x^2 - 3x + \frac{13}{4}$.

The "general quadratic function" is written as $ax^2 + bx + c$. The letters a, b and c stand for certain numbers, such as 1, 0 and 0, or 2, -3 and $\frac{13}{4}$. When they are given specific values, you get a quadratic function of x (assuming that $a \neq 0$). The function is the correspondence between whatever number x you select, and the corresponding number $ax^2 + bx + c$.

This process can continue, using higher and higher powers of x. The functions 1, x, x^2, x^3, x^4, and so forth are called the **power** functions. If we multiply them by "constants," meaning numbers, and add several together, we get a large class of functions, called **polynomial** functions. This includes linear functions (if we use only x and no powers greater than 1), and quadratic functions (if we only use x and x^2). Functions that use only 1, x, x^2, x^3 are sometimes called **cubic** functions, and ones that use up to x^4 are **quartic**, or **bi-quadratic**. Cubic and quartic polynomials were carefully investigated in the Renaissance, but they appear less often in problems, and so are not much studied in high school. There is a little more special terminology beyond quartic, but it is not much used.

Besides polynomial functions in trigonometry or pre-calculus, students may study **trigonometric** functions: the sine function, the cosine function, and certain other functions expressible in terms of these: the tangent function, the cotangent function, the secant function and the cosecant function. These are helpful in dealing with triangles, especially right triangles, in a detailed quantitative way. Trigonometric functions have the striking property of being periodic: they repeat themselves over and over in a systematic way. No polynomials behave like this: polynomials have values that get larger and larger as the size of the variable gets larger.

But even all these functions are far from exhausting the possibilities for functions. Even among numerical functions there are many other kinds. Sometimes we want to talk about functions without being specific about which function we are talking about. The accepted way to do that

is to use a symbol to stand for the function. This is really a very different use of symbols from symbolic algebra, where symbols normally stand for numbers. A function is very much a more complex object than just a number. But in order to keep discussion manageable, we use simple symbolism to represent this rather complicated idea. It is up to you, the reader, to supply the needed interpretation.

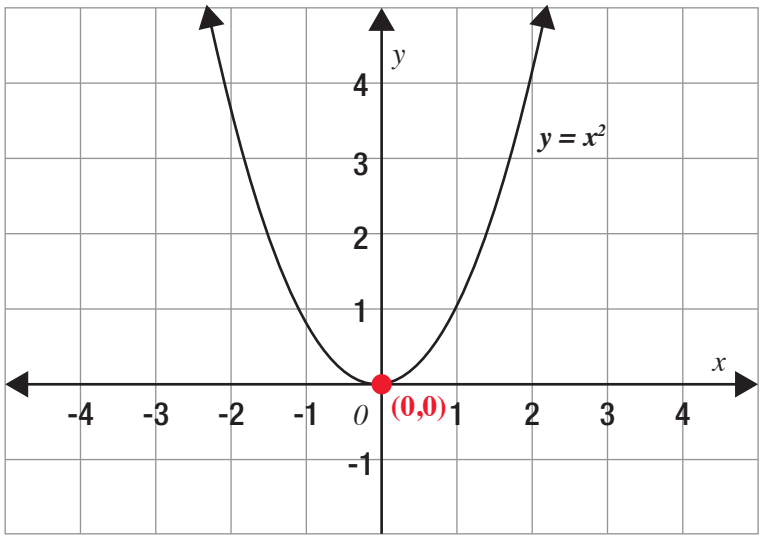
The symbol used to represent a function may be just a single letter, often the letter f. To describe what the function does to a typical input number, we use another letter, such as x, to stand for the input, and then we write $f(x)$ for the output, the number that corresponds to x. We call $f(x)$ the value of f at x. If we want to emphasize the correspondence aspect of the function f, we might indicate it by drawing an arrow:

$$f\colon x \longrightarrow f(x).$$

This should be read as: "the function f takes the number x to the number $f(x)$."

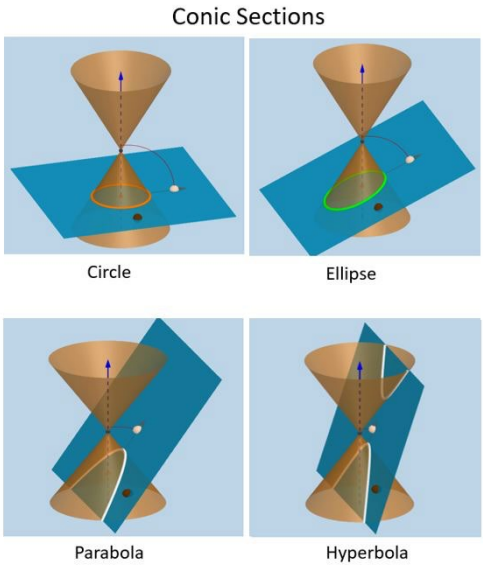
Sometimes you will also see an equation, $y = f(x)$. This indicates that we want to give a separate name to this new number, the result of applying f to x. But it is also the basis for looking at a function from a geometric point of view. Another idea that is heavily used in school mathematics is the coordinate plane (also known as the Cartesian plane, after Rene Descartes, who first proposed using it as a general tool in geometry, and as a link between algebra and geometry). You should know from previous math classes that a typical point in the plane is an ordered pair (x,y) of numbers. The points where $y = f(x)$, that is, the points $(x, f(x))$, give a curve in the plane, with one point on each vertical line whose x coordinate is in the set of inputs. This curve is call the **graph** of f. Looking at the graph of a function gives a lovely connection between algebraic ideas and geometry, and can be very helpful in understanding the behavior of the function. As an example, the graph of any linear function is a straight line in the plane—which of course is why they are called "linear functions." It (almost) goes the other way too: any straight line, except for a strictly vertical one, is the graph of a linear function. This is a very important and wonderful link between geometry and algebra.

Figure 2 - Parabola



Quadratic functions also have a nice geometric interpretation. Their graphs give examples of the kinds of curves called **parabolas** (Figure 1), which belong to the important and attractive family of curves called **conic sections** (Figure 2). Unlike the situation with lines, only a special class of parabolas can be represented as the graphs of quadratic functions. Associated with each parabola, there is a straight line known as the **axis** of the parabola. A parabola is symmetric under reflection across its axis. This can be any line in the plane. If a parabola is described as the graph of a quadratic function, its axis must be a vertical line (and vice versa).

Figure 3 - Conic Sections



But all this discussion has just been about numerical functions. There are huge numbers of these, and they can behave in ways beyond anything we can imagine. But the idea of function is not restricted to numbers. There are even more kinds of other functions that have little or nothing to do with numbers (tThere are some examples below).

So, what is a function? In mathematics, a function is a certain type of correspondence, between collections of "objects" (which can be abstract objects, i.e., ideas, or actual concrete "real world" objects). The formal name for a collection of objects is **set**. There are all kinds of sets, more than we could ever name or even think of. Some examples of sets are:

the set of all people (i.e., human beings)

the set of all whales

the set of all whole numbers

the set of whole numbers larger than 1

the set of all integers

the set of all Americans who voted in the 2016 presidential election

the set of letters of the English alphabet

the set of all positive fractions

the set of all rational numbers

the set of all real numbers

the set of all complex numbers

the set of all circles in the plane

the set of all line segments in the plane

the set of all triangles in the plane.

Correspondences are relationships between members of two sets. A correspondence is often specified by stating a rule. For example, in the set of people the *(biological) parent* correspondence associates each person with his/her biological parents—the people who supplied the DNA that govern the person's cells. This is a correspondence from the set of people to itself. A similar correspondence could be described in the set of whales. In the whole numbers, the **successor** correspondence attaches to each whole number the next larger whole number. It is described by the rule: to get the successor of a number, add 1 to that number. The **prime factor** correspondence associates each whole number greater

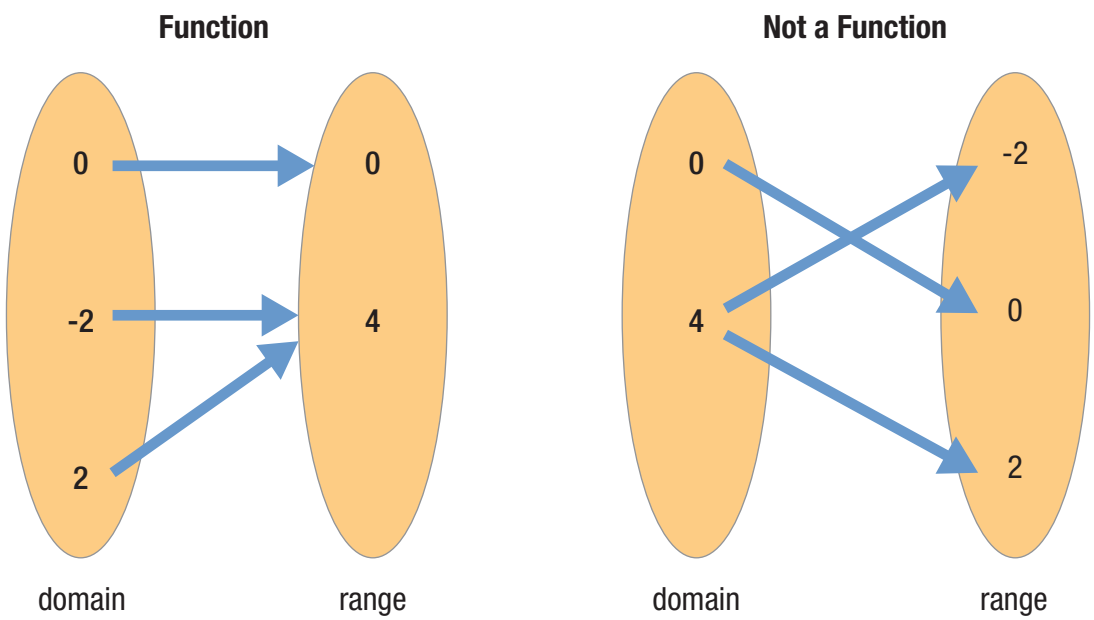
than one with all the prime numbers that exactly divide it. For example, the prime factors of 60 are 2, 3, and 5. The **endpoint correspondence** takes a line segment in the plane to its endpoints, which are a pair of points in the plane. The **vertex correspondence** for triangles takes a triangle in the plane to a triple of points in the plane—the three vertices of the triangle. The **circumcircle correspondence** matches each triangle in the plane to the circle that passes through the three vertices of the triangle. The **radius correspondence** takes a circle in the plane to the number that is the length of its radius.

Maybe these examples are already giving you the idea that a lot of activity in mathematics is devoted to describing correspondences, although the language of functions or correspondences may not be explicitly used. As these examples also show, sometimes the sets involved in a correspondence are the same, and sometimes they are different.

The operations of arithmetic can be thought of in terms of correspondences. Addition of whole numbers gives a correspondence between pairs of whole numbers, and whole numbers: from two whole numbers, you get their sum. Multiplication also gives a correspondence between pairs of whole numbers and whole numbers. Or we could think about addition or multiplication with integers, or rational numbers, or real numbers, or complex numbers, instead of just whole numbers. A lot of arithmetic is devoted to broadening our horizons about what kinds of numbers it makes sense to work with; that is, to enlarging the set of allowable numbers. If we just want to start with a single number, rather than a pair of numbers, we could think about adding a number to itself. For addition, this would give us doubling, or the **doubling correspondence.** If we multiply a number by itself, we call that **squaring**.

The formal definition of **function** is: a function is a correspondence that assigns, to each member of a starting set, called the **domain,** a ***unique*** member of another set, called the **range**. To correctly specify a function, you should start by clearly specifying its domain and its range. But this is often ignored in K–12 mathematics. Some of the examples of correspondences described above are functions, but some are not (Figure 3). Since a person has two parents, the parent correspondence is not a function from people to people (however, it might be refined to the **Mother Correspondence** or the **Father Correspondence**, which are functions). The successor correspondence is a function, but the prime factor correspondence is not, since a number can have (arbitrarily many) prime factors. You might like to go through the other examples and decide which are functions and which are not.

Figure 4 - Example and a non-example of a function



We hope that this note has provided a sense of the pervasive nature of the idea of function in mathematics. Besides the fact that functions can be used to describe many things we do in mathematics, what makes them even more useful is all the things you can do with functions, once you start using them in your thinking. But that is a long story, for another time.

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