THE CIRCLE GAME

In exploring arrangements of tangent circles, the statistician Allan R. Wilks investigated a simple geometric puzzle involving circles packed inside a larger circle:

> *"The route to the discovery began with a high school geometry problem. In 1998, during a lull in a conference in Germany, Wilks started talking with a colleague who was pondering his daughter's homework assignment. The question concerned a pattern made up of two identical circles that fit side by side inside a third circle. The assignment was to find the radius of a fourth circle nestled between the outer circle and the two inner circles.*
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> *After Wilks had returned to the United States, he found the three-circle pattern still on his mind. In drawing it, he could easily insert smaller circles into the empty spaces among the larger circles. Each new circle would be as large as possible without overlapping the circles already present."*
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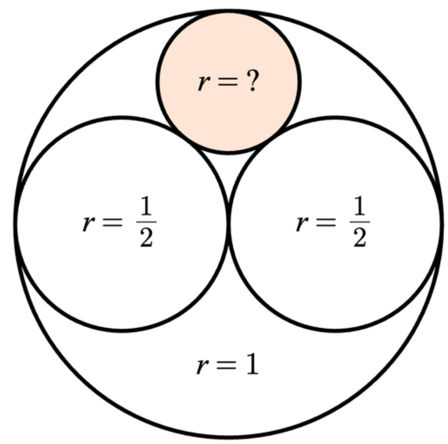
> From: https://www.sciencenews.org/article/circle-game



Find the radius of a fourth circle nestled between the outer circle of radius 1, and the two inner circles, each of radius 1/2.

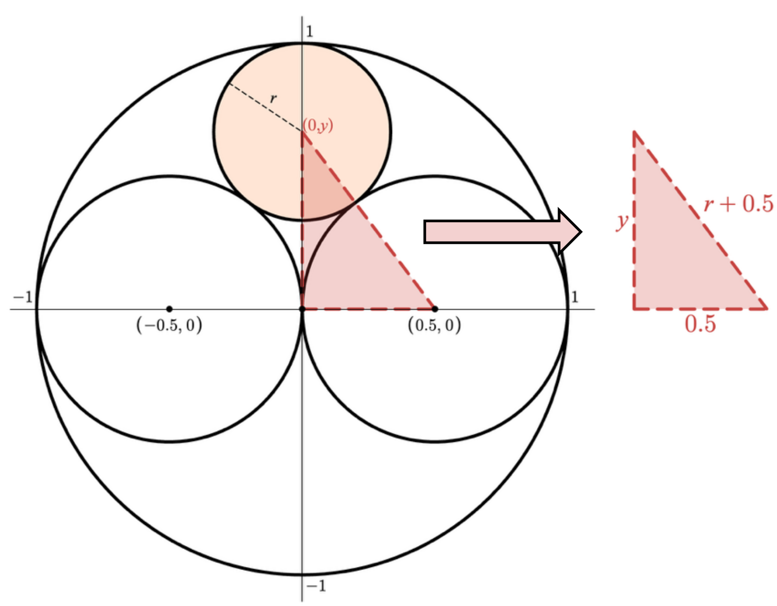
Solution:

Let's assume the larger circle with radius 1 is centered at the origin. The centers of the two inner circles, each with radius $1/2$, can be placed at $(-0.5, 0)$ and $(0.5, 0)$ so that they are tangent to each other at the origin and tangent to the outer circle at $(-1, 0)$ and $(1, 0)$.

Let the fourth circle have an unknown radius r and center $(0, y)$ (by symmetry, its x-coordinate must be 0). This circle must satisfy two conditions:

1. It must be tangent to the outer circle at $(0, 1)$, so

$$y = 1 - r$$



2. It must be tangent to an inner circle, so the distance from its center $(0, y)$ to the center of one inner circle, say $(0.5, 0)$, must be the sum of their radii, $r + 0.5$, and

$$y^2 + 0.5^2 = (r + 0.5)^2$$

by the Pythagorean Theorem.

We can solve this system of equations to find the value of r. Substitute $y = 1 - r$ into $y^2 + 0.5^2 = (r + 0.5)^2$:

$$(1 - r)^2 + 0.5^2 = (r + 0.5)^2$$

$$1 - 2r + r^2 + 0.25 = r^2 + r + 0.25$$

$$1 = 3r$$

$$r = \frac{1}{3}$$

Thus, the radius of the fourth circle is $1/3$.

Additional Context:

Wilks eventually learned that the relationship governing the radii had already been discovered centuries earlier by René Descartes. In 1643, Descartes expressed the relationship in terms of the **curvatures** of the circles, where the curvature of a circle is defined as the reciprocal of its radius. If four mutually tangent circles have curvatures k_1, k_2, k_3, and k_4, then:

$$(k_1^2 + k_2^2 + k_3^2 + k_4^2) = \frac{1}{2}(k_1 + k_2 + k_3 + k_4)^2$$

Let k_1 represent the curvature of the outer circle, then $k_1 = -1/1 = -1$ (because the other circles are *inside* this one, its curvature is considered negative).

Let k_2 and k_3 represent the curvature of the two inner circles with radius $1/2$. Then, $k_2 = k_3 = 2$.

We can substitute the known values into Descartes' formula and solve for k_4:

$$(k_1^2 + k_2^2 + k_3^2 + k_4^2) = \frac{1}{2}(k_1 + k_2 + k_3 + k_4)^2$$

$$((-1)^2 + 2^2 + 2^2 + k_4^2) = \frac{1}{2}(-1 + 2 + 2 + k_4)^2$$

$$(9 + k_4^2) = \frac{1}{2}(3 + k_4)^2$$

$$18 + 2k_4^2 = 9 + 6k_4 + k_4^2$$

$$k_4^2 - 6k_4 + 9 = 0$$

$$(k_4 - 3)^2 = 0$$

$$k_4 = 3$$

Since curvature is the reciprocal of the radius, we have that the radius of the nestled fourth circle is $1/3$.

Mathematician Jeffrey C. Lagarias recognized this configuration as an example of an **Apollonian circle packing**, a geometric construction in which new circles are repeatedly added in the gaps between mutually tangent circles, creating an infinite pattern of nested tangent circles governed by Descartes' curvature formula.

Wilks used a computer to plot the Apollonian packing he was studying. For convenience, he decided to put the origin of his plot at the center of the outer circle and to orient the x-axis so that it passes through the centers of the two inner circles (and in the diagram in the first page). He noticed an unexpected pattern in the coordinates of the circle centers. Although the centers had rational coordinates, multiplying each coordinate by the circle's curvature always produced integers. After Wilks shared this observation with Colin L. Mallows, Mallows proved the relationship and extended it to a formula connecting both the curvatures and the coordinates of four mutually tangent circles. Remarkably, the resulting equation resembles Descartes' curvature formula when the centers are expressed using complex numbers, revealing a deeper algebraic structure within Apollonian circle packings.